Invariants of Affine Weyl Groups

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30 October 2014

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1.1 Set Up

Let $\mathfrak{g}_f$ be a simple finite dimensional Lie algebra over $\mathbb{C}$ of rank l, $\mathfrak{h}_f \subset \mathfrak{g}_f$ a Cartan subalgebra, and $R_f \subset \mathfrak{h}_f^*$ be the root system of $\mathfrak{g}_f$ with respect to $\mathfrak{h}_f$. Fix a root basis $\Pi_f = \{\alpha_i \ (1 \leq i \leq l)\} \subset \mathfrak{h}_f^*$ and the corresponding coroot basis $\Pi^\vee = \{\alpha_i^\vee \ (1 \leq i \leq l)\} \subset \mathfrak{h}_f$. Let

$$P_f = \{\lambda \in \mathfrak{h}_f^* | \lambda(\alpha_i^\vee) \in \mathbb{Z} \ \forall \alpha_i^\vee \in \Pi^\vee\},$$
$$P_f^+ = \{\lambda \in P_f | \lambda(\alpha_i^\vee) \in \mathbb{N} \ \forall \alpha_i^\vee \in \Pi^\vee\}$$

be the weight lattice and the monoid of dominant integral weights. We note the ith fundamental weight by Λ_i, i.e., $\Lambda_i \in P_f$ such that $\Lambda_i(\alpha_j^\vee) = \delta_{i,j}$. We set $\rho = \sum_{i=1}^l \Lambda_i$.

1.2 Group algebra $\mathbb{C}[P_f]$ and their W_f-invariants

We recall some basic facts about the W_f-invariants of the group algebra

$$\mathbb{C}[P_f] = \{\sum_\lambda c_\lambda e^\lambda | c_\lambda = 0 \text{ for all but finite } \lambda\},$$

where the action of W_f is given by $w.e^\lambda := e^{w(\lambda)}$.

To describe a $\mathbb{C}$-basis of $\mathbb{C}[P_f]^{W_f}$, we introduce W_f skew-invariants as follows: for $\lambda \in P_f$, set

$$A_\lambda := \sum_{w \in W_f} \det(w) e^{w(\lambda)}.$$

Fact 1.1. *1.* $W_f(\lambda) \cap P_f^+ \neq \emptyset$.

2. (Skew-symmetry) $A_{w(\lambda)} = \det(w) A_\lambda$.

Hence, we may work with $\lambda \in P_f^+$. Moreover, if $r_{\alpha_i}(\lambda) = 0$ for some $1 \leq i \leq l$, it is clear from the skew-symmetry that $A_\lambda = 0$. Hence, we may restrict ourselves to $\lambda \in P_f^{++} := P_f^+ + \rho$ without loss of generality.

For $\lambda \in P_f^+$, set

$$\chi_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}.$$

It turns out that $\chi_\lambda \in \mathbb{C}[P_f]$ by Weyl's character formula. In particular, we see that

$$\chi_\lambda \text{ has the form } \textstyle\sum_{\alpha \in Q_f^+} c_{\lambda-\alpha} e^{\lambda-\alpha},$$
$$\text{where } Q_f^+ = \mathbb{N}\Pi \text{ is the monoid generated by } \Pi.$$

Hence, we have

Lemma 1.1. $\mathbb{C}[P_f]^{W_f} = \bigoplus_{P_f^+} \mathbb{C}\chi_\lambda$ *as a vector space.*

The ring structure of $\mathbb{C}[P_f]^{W_f}$ can be described as follows. By the fact cited above, it follows that

Theorem 1.2. *1.* χ_{Λ_i} $(1 \leq i \leq l)$ *are algebraically independent and*

2. $\mathbb{C}[P_f^+]^{W_f} = \mathbb{C}[\chi_{\Lambda_1}, \cdots, \chi_{\Lambda_l}]$.

2 Affine Weyl groups

2.1 Set Up

Let $(\cdot, \cdot)$ be the scalar multiple of the bilinear on $\mathfrak{h}_f^*$ induced from the restriction of the Killing form on $\mathfrak{g}_f$ normalized as $(\alpha, \alpha) = 2$ for any long root α. Hence, for any $\alpha \in R_f$, the associated coroot $\alpha^\vee$ is given by

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)}\alpha \in \mathfrak{h}_f^*.$$

Let $\mathrm{Aff}(\mathfrak{h}_f)$ the space of affine linear functions on $\mathfrak{h}_f$ and $L = L(\mathfrak{h}_f) \subset \mathrm{Aff}(\mathfrak{h}_f)$ the subgroup of translations. The actions of L on $\mathfrak{h}_f$ and on $\mathrm{Aff}(\mathfrak{h}_f)$ are given by

$$t(x) := x + t, \qquad (t.\varphi)(x) := \varphi(x - t) \qquad x \in \mathfrak{h},\ \varphi \in \mathrm{Aff}(\mathfrak{h}).$$

For $\varphi \in \mathrm{Aff}(\mathfrak{h}_f)$, we denote by $\widetilde{\varphi}$ the linear part of φ, i.e., $\widetilde{\varphi} \in \mathfrak{h}_f^*$. For a non-constant $\alpha \in \mathrm{Aff}(\mathfrak{h}_f)$, let

$$\pi_\alpha = \{x \in \mathfrak{h}_f | \alpha(x) = 0\}$$

be the hyperplane with respect to α and r_α the reflection with respect to the hyperplane π_α. Denoting $h_\alpha \in \mathfrak{h}_f$ by the normal vector to the hyperplane π_α with the condition $\widetilde{\alpha}(h_\alpha) = 2$, the reflection r_α is defined by

$$r_\alpha(x) = x - \alpha(x)h_\alpha, \qquad r_\alpha(\varphi) = \varphi - \widetilde{\varphi}(h_\alpha)\alpha, \qquad x \in \mathfrak{h}_f, \varphi \in \mathrm{Aff}(\mathfrak{h}_f).$$

Let R_f be an irreducible and reducible finite reduced root system. We define the **affine root system** R_{af} by

$$R_{af} := \{\alpha + l | \alpha \in R_f,\ l \in \mathbb{Z}\}.$$

The group W_{af} generated by r_α ($\alpha \in R_{af}$) is the **affine Weyl group**. It is known that W_{af} isomorphic to the group $W_f \ltimes Q_f^\vee$, where $Q_f^\vee := \mathbb{Z}\Pi^\vee$ is the coroot lattice.

2.2 W_{af}-invariants

We shall identify $\mathfrak{h}_f$ with its dual $\mathfrak{h}_f^*$ via the normalize invariant form as in the preceding subsection.

For $\lambda \in P_f$ and $k \in \mathbb{N}^*$, let $f_{\lambda,k}$ be the function on $\mathbb{H} \times \mathfrak{h}_f \times \mathbb{C}$ defined by

$$f_{\lambda,k}(\tau, z, t) := e^{2\pi\sqrt{-1}(kt+\lambda(z)+\frac{1}{2k}(\lambda,\lambda)\tau)}.$$

We define the action of $Q_f^\vee$ on $\mathbb{H} \times \mathfrak{h}_f \times \mathbb{C}$ by

$$t_\gamma(\tau, z, t) = (\tau, z - \tau\gamma, t - (z, \gamma) + \frac{1}{2}\tau(\gamma, \gamma)).$$

It can be checked that

1. $f_{\lambda,k}(t_\gamma.(\tau, z, t)) = f_{\lambda-k\gamma,k}(\tau, z, t)$, and that
2. a natural action of W_f on $\mathfrak{h}_f$ induces a W_f-action on $\mathbb{H} \times \mathfrak{h}_f \times \mathbb{C}$ with respect to which one has $f_{w(\lambda),k}(\tau, z, t) = f_{\lambda,k}(\tau, w^{-1}(z), t)$.
3. The above two actions define an action of W_{af} on

$$\mathcal{A}_k = \{F \in \mathrm{Hol}(\mathbb{H} \times \mathfrak{h}_f \times \mathbb{C}) | F = \sum_{\lambda \in P_f} c_\lambda f_{\lambda,k} \text{ for some } \{c_\lambda\}_{\lambda \in P_f}\}.$$

Now, for $\lambda \in P_f$ and $k \in \mathbb{N}^*$, we set

$$\theta_{\lambda,k}(\tau, z, t) := \sum_{\gamma \in Q_f^\vee} f_{\lambda+k\gamma,k}(\tau, z, t),$$

which is a (classical) theta function, and

$$A_{\lambda,k}(\tau, z, t) := \sum_{w \in W_f} \det(w)\theta_{w(\lambda),k}(\tau, z, t).$$

Remark 2.1. *Set*

$$C_{af}^k := \left\{ \lambda \in P_f \,\middle|\, \begin{matrix} \lambda(\alpha_i^\vee) \geq 0 \ (1 \leq i \leq l) \\ \lambda(\theta^\vee) \leq k \end{matrix} \right\},$$

where $\theta^\vee$ signifies the highest coroot. It can be checked that for any $\lambda \in P_f$, there exists $\varepsilon \in \{\pm 1\}$ and $\mu \in C_{af}^k$ such that $A_{\lambda,k}(\tau, z, t) = \varepsilon A_{\mu,k}(\tau, z, t)$.

Under these preparations, consider the next problem: $\mathcal{A} := \mathbb{C} \oplus \bigoplus_{k \in \mathbb{N}^*} \mathcal{A}_k$.

Study the structure of the algebra of invariants $\mathcal{A}^{W_{af}}$.

Well, the easy part is as follows. Let $h^\vee := 1+\sum_{i=1}^{l} a_i^\vee$, where $\sum_{i=1}^{l} a_i^\vee \alpha_i^\vee = \theta^\vee$, be the dual Coxeter number. For $k \in \mathbb{N}^*$ and $\lambda \in C_{af}^k$ we set

$$\chi_{\lambda,k}(\tau,z,t) := \frac{A_{\lambda+\rho,k+h^\vee}(\tau,z,t)}{A_{\rho,h^\vee}(\tau,z,t)}.$$

Then, a simple calculation shows

Lemma 2.1. $\mathcal{A}^{W_{af}} = \mathbb{C} \oplus \bigoplus_{k\in\mathbb{N}^*} \bigoplus_{\lambda\in C_{af}^k} \mathbb{C}\chi_{\lambda,k}$.

We remark that

1. this ring $\mathcal{A}^{W_{af}}$ has a structure of graded ring graded by $k \in \mathbb{N}^*$, and
2. as a graded vector space, $\mathcal{A}^{W_{af}}$ is isomorphic to $\mathbb{C}[X_0,\cdots,X_l]$ where the degree of each variable X_i is defined by $a_i^\vee$.

Here, $a_0^\vee := 1$ by definition. So, a natural question is

Can an isomorphism $\mathcal{A}^{W_{af}} \cong \mathbb{C}[X_0,\cdots,X_l]$ above as graded vector spaces be indeed an isomorphism as graded $\mathbb{C}$-algebras ?

2.3 History : Algebraic structure of $\mathcal{A}^{W_{af}}$

In 1976, E. Looijenga [L] 'showed' that

for any $\tau \in \mathbb{H}$, the response to the above question is "YES"

with erreur !

In 1978, I. N. Bernstein and O. Schwarzman [BS] pointed out the erreur and announced the result in full generality with their 'sketch of proof' only in 2 pages.

By computing explicitly the Jacobian of the fundamental characters $\{\chi_{0,1}, \chi_{\Lambda_i,a_i^\vee}\ (1 \le i \le l)\}$, D. H. Peterson 'showed' that the response to the above question is "YES" for any $\tau \in \mathbb{H}$ in the case when R_f is of type A_l, B_l, C_l, D_l and of type G_2. This result was announced in the paper with V. Kac [KP] in Appendix 4 (Section 4.10) saying that `a detail will appear elsewhere`, which in fact never appears.

Nearly 20 years after their publication of the announcement [BS], J. Bernstein and O. Schwarzman in [BS2] published a detailed version of [BS] excluding type $D_l^{(1)}$ with the excuse saying that this case is covered by a result of D. H. Peterson !

Hence, even now, there is no published proof for this case.......

Remark 2.2. *J. Bernstein and O. Schwarzian [BS2] treated also twisted cases, except for type* $A_{2l}^{(2)}$ *which had been covered by the computation dur to D. H. Peterson.*

2.4 Application

This result is used when one discuss about the moduli space of semi-stable G-bundles over a torus $E_\tau := \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, where G is a connected and simply-connected simple algebraic group over $\mathbb{C}$. Indeed, one can show that such space can be expressed as the quotient

$$E_\tau \otimes_{\mathbb{Z}} Q_f^\vee / W_f$$

where the group W_f acts on the right component, i.e., $Q_f^\vee$. By the (T)HOEREM, one sees that this space is isomorphic to the weighted projective space

$$\mathbb{P}(a_0^\vee, a_1^\vee, \cdots, a_l^\vee).$$

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