The space of connection data of q-linear equations and q-Painlevé equations

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Abstract : A q-analogue of the Painlevé equation can be obtained by connection preserving deformations. Jimbo and Sakai found a q-analogue of the sixth Painlevé equations by connection preserving deformations of a q-difference equations. We study the space of connection data of q-linear equations, which is a q-analogue of the Fricke cubic. We give a connection between connection data and a solution of q-Painlevé equations.

Y Ohyama, J.-P. Ramis, J. Sauloy, `arXiv:2005.10122`

The space of monodromy data for the Jimbo-Sakai family of q-difference equations.

1. Painlevé equations, isomonodromy, asymptotics

The Painlevé equations are obtained by isomonodromic deformations

- **The 'moduli' space of connections** gives an initial values spaces
- **The space of monodromy/Stokes data** becomes a cubic surface (**Fricke**, ..., Saito-van der Put)
- **The Riemann-Hilbert correspondence**
- **Asymptotic expansion of Painlevé transcendents** gives a monodromy Stokes parameters (**Jimbo**, Kapaev, ...)

The sixth Painlevé equation $\mathrm{P_{VI}}$

$$y'' = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right){y'}^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right]$$

is given by **isomonodromy deformation** of a 2×2 system.

1.1 Isomonodromy deformation

$$\frac{\partial Y}{\partial x} = \left(\frac{A_0(t)}{x} + \frac{A_t(t)}{x-t} + \frac{A_1(t)}{x-1}\right) Y \equiv A(x,t)Y$$

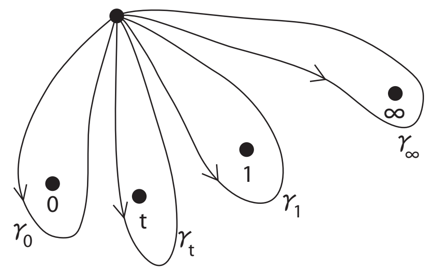
$$\frac{\partial Y}{\partial t} = -\frac{A_t(t)}{x-t} Y$$

Theorem We set $A_{12} = k(x - y(t))$. $y(t)$ satisfies P_{VI} for

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2,\ \beta = -\frac{1}{2}(\theta_0 - 1)^2, \gamma = \frac{1}{2}(\theta_1 - 1)^2, \delta = \frac{1}{2}(1 - \theta_t^2).$$

$A_j \in SL(2,\mathbb{C})$, the eigenvalues of A_j : $\pm\frac{1}{2}\theta_j$.

$$A_0(t) + A_t(t) + A_1(t) = -\frac{1}{2}\begin{pmatrix}\theta_\infty & \\ & -\theta_\infty\end{pmatrix}$$



The monodromy matrices M_0, M_t, M_1, M_∞ satisfy

$$M_\infty M_1 M_t M_0 = 1.$$

1.2 Jimbo's P6 asymptotics

Known asymptotics of equations are related to the linear monodromy data:

Example: **Jimbo's sol** for **P6** [Publ. RIMS, **18** (1982).]

$$y(t) = \sum_{n=1}^{\infty} t^n \sum_{m=-n}^{n} c_{nm} t^{m\sigma}$$

Here c_{nm} is written by the Gamma function on p_{jk} and

$$\sigma = \frac{1}{\pi}\arccos\left(\frac{p_{0t}}{2}\right), \quad p_{jk} = \mathrm{tr}\, M_j M_k \ (j,k = 0,1,t)$$

Monodromy invariants

$$p_\nu := \mathrm{tr}\ M_\nu = 2\cos 2\theta_\nu, \quad p_{\mu\nu} := \mathrm{tr}\ M_\mu M_\nu.$$

The Fricke relation:

$$p_{01}p_{1t}p_{t0} + p_{01}^2 + p_{1t}^2 + p_{t0}^2 - a_{01}p_{01} - a_{1t}p_{1t} - a_{t0}p_{t0} + a_\infty = 0.$$

$$a_{ij} = p_i p_j + p_k p_l \quad (\{i,j,k,l\} = \{0,t,1,\infty\}),$$
$$a_\infty = p_0^2 + p_1^2 + p_t^2 + p_\infty^2 + p_0 p_1 p_t p_\infty - 4.$$

1.3 Other Painlevé equations

All linear equations are **2x2 systems**

P1, P2 $$\frac{dY}{dx} = \left(A + Bx + Cx^2\right) Y(x)$$

♣ C is nilpotent for P1

P4 $$\frac{dY}{dx} = \left(\frac{A}{x} + B + Cx\right) Y(x)$$

♣ when C is nilpotent, P34

P3 $$\frac{dY}{dx} = \left(\frac{A}{x^2} + \frac{B}{x} + Cx\right) Y(x)$$

♣ A, C may nilpotent for D_7, D_8

P5 $$\frac{dY}{dx} = \left(\frac{A}{x} + \frac{B}{x-1} + C\right) Y(x)$$

♣ when C is nilpotent, deg-P5 $\sim$ P3

P6 $$\frac{dY}{dx} = \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-t}\right) Y(x)$$

1.4 P1: $y'' = 6y^2 + t$

Isomonodromy deformation $\dfrac{dY}{dz} = A(z,t)Y$

$$A(z,t) = \begin{pmatrix} 4z^4 + t + 2y^2 & 4yz^2 + t + 2y^2 \\ -(4yz^2 + t + 2y^2) & -(4z^4 + t + 2y^2) \end{pmatrix} - (2y'z + \frac{1}{2z})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y(z) \sim \exp\left[(\frac{4}{5}z^5 + tz) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

as $|z| \to \infty$, $\arg z \in \left(\frac{\pi}{5}(k - \frac{3}{2}), \frac{\pi}{5}(k + \frac{1}{2}) \right)$.

Stokes data

$$S_k = Y_k^{-1}(z) Y_{k+1}(z)$$

The cyclic relation: $S_1 S_2 S_3 S_4 S_5 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. If $1 + s_2 s_3 \neq 0$,

$$s_1 = \frac{i - s_3}{1 + s_2 s_3}, \quad s_4 = \frac{i - s_2}{1 + s_2 s_3}, \quad s_5 = i(1 + s_2 s_3)$$

- If $1 + s_2 s_3 = 0$, $s_2 = s_3 = i$, $s_5 = 0$, $s_1 + s_4 = i$.
- For real solution (y, z are real when t is real) $s_2 = -\bar{s}_3$.

1.4.1 Boutroux transformation of P1: $y'' = 6y^2 + t$

$$y = (e^{-\pi i}t)^{1/2}v, \quad x = \frac{4}{5}(e^{-\pi i}t)^{5/4}$$

P1 is written in '**almost elliptic**' form

$$\mathbf{v'' = 6v^2 + 1} + \frac{4v}{25x^2} - \frac{v'}{x},$$

$$v \sim -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{5x}}\left[a\exp(-ih) + b\exp(ih)\right],$$

$$a = \left(\frac{2}{3}\right)^{1/8}\frac{\sqrt{2}\pi}{s_2\Gamma(-i\rho)}\exp\frac{\pi\rho}{2}, \quad b = -\left(\frac{2}{3}\right)^{1/8}\frac{\sqrt{2}\pi}{s_3\Gamma(i\rho)}\exp\frac{\pi\rho}{2}.$$

$$h = 2\left(\frac{2}{3}\right)^{1/4}x + \rho\log 5x + \frac{11}{4}\rho\log 2 + \frac{5}{4}\rho\log 3 + \frac{3\pi}{4},$$

$$\rho = \frac{1}{2\pi}\log(1 + s_2 s_3),$$

Remark By the Boutroux transformation, v is **irregular singular of the Poincaré rank 1** at $x = \infty$

1.5 The Riemann-Hilbert Problem

The Riemann-Hilbert correspondence

$$\{\text{ The space of connections }\} \Longrightarrow \{\text{ The space of monodromy data }\}$$

is highly transcendental.

1) We know the Riemann-Hilbert correspondence is **one-to-one**, if we set the both spaces suitably.

2) When the linear equation is **rigid**, we can determine monodromy/Stokes data algebraically.

3) When the linear equation with **accessary parameters** has isomonodromy deformations, we can calculate monodromy/Stokes data by **asymptotic analysis**.

Remark. additive E_6, E_7, E_8 Painlevé equations do not have continuous deformations.

Problem : How about q-analogue ?

2. q-P$_{\mathrm{VI}}$ by Jimbo-Sakai

A **q-analogue of the sixth Painlevé equation**

$$\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z}-b_1t)(\bar{z}-b_2t)}{(\bar{z}-b_3)(\bar{z}-b_4)}, \qquad \frac{z\bar{z}}{b_3b_4} = \frac{(y-a_1t)(y-a_2t)}{(y-a_3)(y-a_4)},$$

$$\frac{b_1b_2}{b_3b_4} = q\frac{a_1a_2}{a_3a_4}.$$

Connection Preserving Deformation (CPD)

2×2 system of q-difference equation ($0<|q|<1$)

$$Y(qx,t) = A(x,t)Y(x,t) = [A_0 + xA_1 + A_2x^2]Y(x,t),$$
$$Y(x,qt) = B(x,t)Y(x,t).$$

$$B(x,t) = \frac{x}{(x-a_1qt)(x-a_2qt)}(xI + B_0(t)).$$

The compatibility: $\quad A(x,qt)B(x,t) = B(qx,t)A(x,t).$

- The **eigenvalues** of A_0 is ρ_1 and ρ_2.
- $A_2 = \mathrm{diag}(\kappa_1,\kappa_2)$
- $\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)(x-a_4).$

2.1 Connection data

The **connection matrix** $P(x)$ is

$$Y_\infty(x,t) = Y_0(x,t)P(x,t)$$

and $P(x,t)$ is a **q-constant**

$$P(xq,t) = P(x,t), \quad P(x,tq) = P(x,t).$$

Although $P(x)$ is not a constant matrix, P contains a finite number of parameters.

G. D. Birkhoff studied a q-analogue of Riemann-Hilbert problem (**Riemann-Hilbert-Birkhoff correspondence**)

Consider **the space of connection matrices**

Study **asymptotic expansion of q-P_{VI}**

Remark. The asymptotic expansion is studied by Mano.

2.2. Basic notations

0) **q-shifted factorial**:

$$(a_1, \ldots, a_r; q)_n = \prod_{i=1}^{n} (a_i; q)_n, \quad (a; q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a).$$

1) **Theta function**:

$$\theta_q(x) := \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} x^n = (q, -x, -q/x; q)_\infty.$$

$$e_c(x) := \frac{\theta(x)}{\theta(cx)}, \text{ for } c \in \mathbb{C}^\times$$

First order difference equation:

$$x\theta_q(xq) = \theta_q(x), \quad e_c(xq) = ce_c(x), \quad (1-ax)(axq;q)_\infty = (ax;q)_\infty$$

2) **generalized q-hypergeometric series**:

$$\begin{aligned} &{}_r\varphi_s\left(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z\right) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \end{aligned}$$

2.2. Birkhoff's Riemann-Hilbert correspondence

We start from a **q-difference equation** of size r:

$$Y(qx) = A(x)Y(x),$$

$$A(x) = A_0 + xA_1 + \cdots + x^N A_N$$

We assume that the **eigenvalues** of A_0 is $\rho_1, ..., \rho_r (\neq 0)$ **[regular singular]** and **non-resonance condition**

$$\rho_j/\rho_k \notin q^{\mathbb{Z}}$$

if $j \neq k$. We also assume that

$$A_N = \begin{bmatrix} \kappa_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \kappa_r \end{bmatrix}.$$

$\kappa_1, ..., \kappa_r (\neq 0)$ and $\kappa_j/\kappa_k \notin q^{\mathbb{Z}}$ $(j \neq k)$.

We assume that $\det A(x)$ has simple zeros $a_1, ..., a_{rN}$ s.t. $a_j \neq a_k$ $(j \neq k)$:

$$\det A(x) = \kappa_1\kappa_2 \cdots \kappa_r (x - a_1)(x - a_2) \cdots (x - a_{rN})$$

Local solutions around $x = 0$ and $x = \infty$:

$$Y_0(x) = L(x) \begin{bmatrix} e_{\rho_1}(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_{\rho_r}(x) \end{bmatrix},$$

and

$$Y_\infty(x) = \theta(x)^{-N} R(x) \begin{bmatrix} e_{\kappa_1}(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_{\kappa_r}(x) \end{bmatrix}.$$

Here

$$L(x) = \sum_{j=0}^{\infty} L_j x^j, \quad R(x) = \sum_{j=0}^{\infty} R_j x^{-j}.$$

$\mathrm{diag}(\rho_1, ..., \rho_r) = L_0^{-1} A_0 L_0$.
In general, we assume that $R_0 = I_r$ and $\det L_0 = 1$.

Definition 1. We define the **connection matrix** $P(x)$ as

$$Y_\infty(x) = Y_0(x) P(x).$$

⋆ In the q-difference case, $P(x)$ **is an elliptic function**, not constants.

We obtain

$$P(x) = \theta(x)^{-N} \begin{bmatrix} e_{\rho_1}(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_{\rho_r}(x) \end{bmatrix}^{-1} L(x)^{-1} R(x) \begin{bmatrix} e_{\kappa_1}(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_{\kappa_r}(x) \end{bmatrix}.$$

$L(x)^{-1}$ and $R(x)$ is holomorphic on $\mathbb{C}^\times$ and all of matrix elements of $P(x)$ are elliptic:

$$P(xq) = P(x), \quad P(xe^{2\pi i}) = P(x).$$

Therefore $L(x)^{-1}R(x)$ are **holomorphic of degree N**.
We set $[p_{ij}(x)] = L(x)^{-1}R(x)$. Then we have

$$p_{ij}(xe^{2\pi i}) = p_{ij}(x), \quad p_{ij}(xq) = x^N \frac{\kappa_j}{\rho_i} p_{ij}(x).$$

Lemma 2.

$$p_{ij}(x) = p_{ij}^\circ \prod_{k=1}^{N} \theta(x/c_{ij}^{(k)}),$$

where

$$\prod_{k=1}^{N} c_{ij}^{(k)} \frac{\kappa_j}{\rho_i} = 1.$$

The set of Birkhoff's parameters is

$$\{p^{\circ}_{ij};\ c^{(k)}_{ij}\}.$$

The order of the set is $r^2 + (N-1)r^2$.
Y_0 and Y_∞ have **ambiguity of the right action of diagonal matrices**.

Theorem 3. **(Birkhoff, 1914)** *The number of essential parameters of the connecton matrix $P(x)$ is*

$$[r^2 + (N-1)r^2] - (2r-1) = Nr^2 - 2r + 1.$$

The number of parameters of the equation $A(x)$ is

$$Nr^2 - r.$$

But $A(x)$ also admits adjoint action by a diagonal matrix, **The number of essential parameters of the equation $A(x)$** is

$$[Nr^2 - r] - (r-1) = Nr^2 - 2r + 1.$$

2.3. Examples

1) When $r=1$, $Nr^2-2r+1=N-1$. (**1st order difference equation**).
This parameter corresponds to $a_1\cdots a_{N-1}a_N=\rho_1\kappa_1^{-1}$.

2) When $r=2, N=1$, $Nr^2-2r+1=1$ (**basic hypergeometric function**).

$$Y(qx)=(A_0+A_1x)Y(x)$$

$A_1=\mathrm{diag}(\kappa_1,\kappa_2),\det(A_0+A_1x)=\kappa_1\kappa_2(x-a_1)(x-a_2)$.

$$A_0=\begin{pmatrix}-\kappa_1\alpha & \kappa_2 w\\ \kappa_1 w^{-1} & \gamma-\kappa_2\beta\end{pmatrix}$$

In above two cases, the connection matrix is completely determined by exponents. In this sense, they are **rigid** systems.

3) When $r=2, N=2$, $Nr^2-2r+1=5$ (q-**P**$_{\mathrm{VI}}$).
This parameter corresponds to $a_1a_2a_3a_4=\rho_1\rho_2(\kappa_1\kappa_2)^{-1}$ plus two more parameters (**accessary parameters**).

2.4. Basic hypergeometric series

Basic hypergeometric series ${}_2\varphi_1(a,b;c;x)$:

$$(c-abqx)u(xq^2)-[c+q-(a+b)qx]u(qx)+q(1-x)u(x)=0.$$

Local solutions around $x=0$:

$$u_1={}_2\varphi_1\left(a,b;c;x\right),\quad u_2=e_{q/c}(x){}_2\varphi_1\left(qa/c,qb/c;q^2/c;x\right).$$

Local solutions around $x=\infty$:

$$v_1=\frac{1}{e_a(-x)}{}_2\varphi_1\left(a,aq/c;aq/b;cq/abx\right),\ v_2=(a\leftrightarrow b).$$

Theorem (Thomae) Connection formula for ${}_2\varphi_1$:

$$u_1=\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}v_1+\frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}v_2,$$
$$u_2=\frac{(qb/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{e_a(-x)e_{q/c}(x)}{e_{qa/c}(-x)}v_1+\frac{(qa/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{e_b(-x)e_{q/c}(x)}{e_{qb/c}(-x)}v_2.$$

2.4.1. Another method

We set $A_1 = \mathrm{diag}(\kappa_1, \kappa_2), \det(A_0 + A_1 x) = \kappa_1\kappa_2(x - a_1)(x - a_2)$.

$$Y(qx) = (A_0 + A_1 x)Y(x)$$

The **connection matrix** is given by

$$P(x) = \theta(x)^{-1} \begin{bmatrix} e_{\rho_1}(x) & 0 \\ 0 & e_{\rho_2}(x) \end{bmatrix}^{-1} \begin{bmatrix} p^\circ_{11}\theta(\kappa_1 x/\rho_1) & p^\circ_{12}\theta(\kappa_2 x/\rho_1) \\ p^\circ_{21}\theta(\kappa_1 x/\rho_2) & p^\circ_{22}\theta(\kappa_2 x/\rho_2) \end{bmatrix} \begin{bmatrix} e_{\kappa_1}(x) & 0 \\ 0 & e_{\kappa_2}(x) \end{bmatrix}$$

Since

$$\det P(x) = \frac{e_{\kappa_1}(x)e_{\kappa_2}(x)}{\theta(x)^2 e_{\rho_1}(x)e_{\rho_2}(x)e_{-1/a_1}(x)e_{-1/a_2}(x)},$$

$$p^\circ_{11}p^\circ_{22}\theta\left(\frac{\kappa_1 x}{\rho_1}\right)\theta\left(\frac{\kappa_2 x}{\rho_2}\right) - p^\circ_{12}p^\circ_{21}\theta\left(\frac{\kappa_2 x}{\rho_1}\right)\theta\left(\frac{\kappa_1 x}{\rho_2}\right) = \frac{1}{e_{-1/a_1}(x)e_{-1/a_2}(x)}.$$

Since $e_{-1/a_1}(a_1) = \infty$ and the inversion formula $x\theta(1/x) = \theta(x)$,

$$\frac{p^\circ_{11}p^\circ_{22}}{p^\circ_{12}p^\circ_{21}} = \frac{\theta(\kappa_2 a_1/\rho_1)\theta(\kappa_1 a_1/\rho_2)}{\theta(\kappa_1 a_1/\rho_1)\theta(\kappa_2 a_1/\rho_2)} = \frac{\theta(\kappa_2 a_2/\rho_1)\theta(\kappa_1 a_2/\rho_2)}{\theta(\kappa_1 a_2/\rho_1)\theta(\kappa_2 a_2/\rho_2)}.$$

Constants p°_{ij} **are determined**, up to diagonal actions from the both side.

3.1 Double asymptotic solutions of q-Painlevé

This type of expansions are studied by **R. Fuchs** at first.

Jimbo also gives the same type of asymptotic solutions for $P_{\rm III}$, $P_{\rm V}$ and $P_{\rm VI}$. He also showed that the double asymptotic series **converges in a small angle domain**.

Mano studied the case of $P(A_3)$ (q-$P_{\rm VI}$).

Connection formula of q-$P_{\rm VI}$

$$Y(qx,t) = A(x,t)Y(x,t) = [A_0 + xA_1 + A_2x^2]Y(x,t),$$

1. **The first limit**

We take a limit $t \to 0$. Then $A(x,t)$ goes to $x\mathbf{\Lambda} + x^2A_2$

2. **The second limit**

We set $x = \xi t$. We take a new connection:

$$\tilde{A}(\xi,t) = t^{-1}t^{-\log_q \Lambda}A(t\xi,t)t^{\log_q \Lambda}$$

We take a limit $t \to 0$. Then $\tilde{A}(\xi,t)$ goes to $M + \xi\mathbf{\Lambda}$. $M \sim A_0/t$

The limit equations

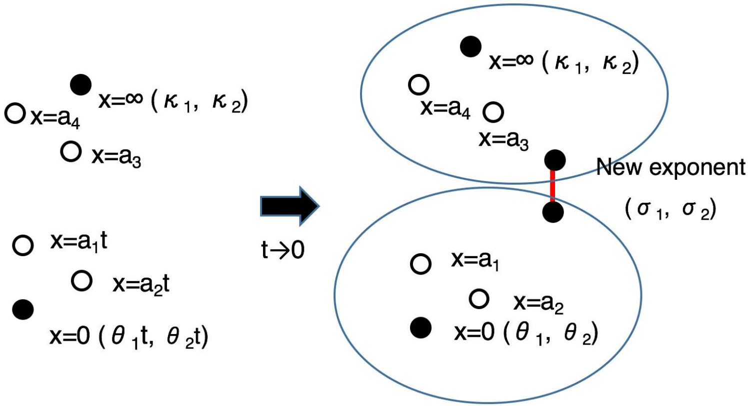
$$Y_1(xq) = [x(\Lambda + xA_2)]Y_1(x),\ Y_2(\xi q) = (M + \xi\Lambda)Y_2(\xi)$$

are reduced to hyperegeometric.

We set eigenvalues of Λ as σ_1, σ_2. (assume that $\sigma_1\sigma_2 \neq 0$)

$$C^{-1}\Lambda C = \mathrm{diag}(\sigma_1, \sigma_2).$$

We write $D = \mathrm{diag}(\log_q \sigma_1, \log_q \sigma_2)$. Then $t^{\log_q \Lambda} = C^{-1}t^D C$.



Theorem (Mano) $\quad P(x) = P_2(x/t)P_1(x)$

3.2 How to calculate connection matrix

1) **The original equation**:

$$Y^{(\infty)}(xq,t) = A(x,t)Y^{(\infty)}(x,t)$$

$$Y^{(\infty)}(x,t) = q^{u(u-1)}\hat{Y}^{(\infty)}(x,t)x^K, \ \ (u = \log x, \hat{Y}^{(\infty)}(0) = I)$$

2) **The first limit of solution**: From $Y^{(\infty)}(x,t)$ to $Y_1(x)$:

$$\lim_{t\to 0} Y^{(\infty)}(x,t) =: Y_1^{(\infty)}(x).$$

$$Y_1^{(\infty)}(xq) = [x(\Lambda + xA_2)]Y_1^{(\infty)}(x)$$

3) **Connection formula** for $Y_1(x)$:

$$Y_1^{(\infty)}(x) = Y_1^{(0)}(x)P_1(x),$$

$$Y_1^{(0)}(x) = q^{u(u-1)/2}\mathbf{C}\hat{Y}^{(0)}(x)x^D, \ \ (\hat{Y}_0(0) = I)$$

where $C^{-1}\Lambda C = \mathrm{diag}(\sigma_1, \sigma_2)$, and $D = \mathrm{diag}(\log_q \sigma_1, \log_q \sigma_2)$.
C has an **anbiguity** $C \to CG$, G is a diagonal matrix.

4) **The second limit** : Set $\xi = x/t$.

$$Y_2^{(\infty)}(\xi q) = (M + \xi\Lambda) Y_2^{(\infty)}(\xi)$$

$$Y_2^{(\infty)}(\xi) = q^{v(v-1)/2} \mathbf{C} \hat{Y}_2^{(\infty)}(x) x^D.$$

5) **Connection formula** for $Y_1(x)$

$$Y_2^{(\infty)}(\xi) = Y_2^{(0)}(\xi) P_2(\xi)$$

6) We can reconstruct $Y(x,t)$ from Y_1 by an **iteration operator**

$$U(x,t) = I + \sum_{n=1}^{\infty} \int_0^t d_q t_1 \int_0^{t_1} d_q t_2 \cdots \int_0^{t_{k-1}} d_q t_k F(x,t_1) F(x,t_2) \cdots F(x,t_k).$$

Here

$$F(x,t) = \frac{1}{t(q-1)} (B(x,t) - I),$$

$$B(x,t) = I + \frac{1}{x} \left(1 - \frac{qta_1}{x}\right) \left(1 - \frac{qta_2}{x}\right) [B_0(t) + qt(a_1 + a_2) I - x^{-1} q^2 t^2 a_1 a_2 I]$$

Lemma.

$$Y^{(\infty)}(x,t) = U(x,t)Y_1^{(\infty)}(x),$$

$$Y^{(0)}(x,t)P_2(x/t) = U(x,t)Y_1^{(0)}(x).$$

Proof is the standard way to solve an **integral equation** starting from the initial value $Y_1^{(\infty)}(x)$.

Proof of Theorem:
By the Lemma above, we have

$$\begin{aligned} Y^{(0)}(x,t)P_2(x/t)P_1(x) &= U(x,t)\underline{Y_1^{(0)}(x)P_1(x)} \\ &= U(x,t)\underline{Y_1^{(\infty)}(x)} \\ &= Y^{(\infty)}(x,t). \end{aligned}$$

Therefore

$$Y_1^{(\infty)}(x) = Y^{(0)}(x,t)P_2(x/t)P_1(x) = Y^{(0)}(x,t)P(x).$$

4. The space of connection matrices

We start

$$Y(qx) = [A_0 + xA_1 + \cdots + x^N A_N]Y(x),$$

$$\det A(x) = \kappa_1\kappa_2\cdots\kappa_r(x-a_1)(x-a_2)\cdots(x-a_{rN})$$

Set $y(x) = \det Y(x)$. Then $y(x)$ is represented by theta functions:

$$y(xq) = \kappa_1\kappa_2\cdots\kappa_r(x-a_1)(x-a_2)\cdots(x-a_{rN})y(x).$$

Local solutions:

$$Y_0(x) = L(x)\,\mathrm{diag}\{e_{\rho_1}(x), ..., e_{\rho_r}(x)\},$$

$$Y_\infty(x) = \theta(x)^{-N}R(x)\,\mathrm{diag}\{e_{\kappa_1}(x), ..., e_{\kappa_r}(x)\}.$$

Proposition (**Birkhoff factorisation**)
(1) $L(x)^{-1}$ is holomorphic on $\mathbb{C}^\times$. $L(x)$ has simple poles over $q^{-\mathbb{N}}a_j$ $(j = 1, 2, ..., r)$.
(2) $R(x)$ is holomorphic on $\mathbb{C}^\times$. $R(x)^{-1}$ has simple poles over $q^{\mathbb{Z}+}a_j$ $(j = 1, 2, ..., r)$.

4.1 The space of connection matrices (Suite)

Therefore

$$M = M(x) = L(x)^{-1}R(x)$$

is holomorphic on $\mathbb{C}^\times$. and M^{-1} has simple poles over $q^{\mathbb{Z}}a_j$

Proposition

$$\sigma_q M = x^{-N} R M S^{-1}$$

Here

$$R = \mathrm{diag}(\rho_1, ..., \rho_r), \quad S = (\kappa_1, ..., \kappa_r).$$

$$F_{R,S,\underline{a}} = \{M \,|\, \sigma_q M = x^{-N} R M S^{-1}, \det M(a_j) = 0 \text{ (simple)}\}$$

Since $Y_0(x)$ and $Y_\infty(x)$ has a **gauge freedom**

$$Y_0(x) \sim Y_0(x)\Gamma, \; Y_\infty(x) \sim Y_\infty(x)\Delta,$$

where Γ and Δ are constant diagonal matrices. Therefore

$$M' \sim \Gamma^{-1} M \Delta$$

4.2 The space of connection matrices (Suite)

Definition The **space of connection matrices**

$$\mathcal{F}_{R,S,\underline{a}} = \{M \mid \sigma_q M = x^{-N} RMS^{-1}, \det M(a_j) = 0 \text{ (simple)}\}/\sim$$

We consider a q-difference equation

$$Y(xq) = A(x)Y(x)$$

here $A(x) = A_0 + xA_1 + \cdots + x^N A_N$ with non-resonance condition.

Definition $A \sim B$ if and only if there exists $F \in GL_r(\mathbb{C}(x))$ such that

$$B = (\sigma_q F)AF^{-1}$$

Definition The **space of q-difference equations**

$$\mathcal{E}_{R,S,\underline{a}} = \{A(x) \mid A(x) = A_0 + xA_1 + \cdots + x^N A_N\}/\sim$$

Theorem (?) The map

$$\mathcal{E}_{R,S,\underline{a}} \to \mathcal{F}_{R,S,\underline{a}}$$

is onto and one-to one.

4.3 Sketch of proof

1. Injectivity

Let $A \in E_{R,S,\underline{a}}$, resp. $B \in E_{R,S,\underline{a}}$,

$$M_A = L_A^{-1} R_A, \quad M_B = L_B^{-1} R_B$$

Since $M_A = \Gamma^{-1} M_B \Delta$,

$$L_A^{-1} R_A = \Gamma^{-1} L_B^{-1} R_B \Delta \Longrightarrow L_B \Gamma L_A^{-1} = R_B \Delta R_A^{-1}$$

If w e set $F = L_B \Gamma L_A^{-1} = R_B \Delta R_A^{-1}$. Then

$$B = (\sigma_q F) A F^{-1}.$$

2. Surjectivity

Based on **Birkhoff's factorization theorem**:

Let C a simple closed analytic curve on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, separating 0 from ∞ and $\mathbb{P}^1(\mathbb{C}) \setminus C = D_0 \cup D_\infty$.

Let $M(x)$ a given analytic invertible matrix on C.

Then there exist analytic matrices M_0 on $\overline{D_0}$ and M_∞ on $\overline{D_\infty}$ such that $M_0 = M_\infty M$ on C.

5 Structure of $\mathcal{F}$: q-P$_{\rm VI}$

Jimbo-Sakai's q-$P_{\rm VI}$

$$Y(qx) = A(x)Y(x) = [A_0 + xA_1 + A_2x^2]Y(x),$$

$$R = \mathrm{diag}(\rho_1, \rho_2), \quad S = \mathrm{diag}(\kappa_1, \kappa_2), \quad A_0 = \Gamma R \Gamma^{-1}, \quad A_2 = \Delta S \Delta^{-1}$$

$$\det A(x,t) = \kappa_1\kappa_2(x-a_1)(x-a_2)(x-a_3)(x-a_4).$$

$$L(xq) = A(x)L(x)R^{-1}, \quad L(x) = \Gamma + \cdots \in GL_2(\mathbb{C}\{x\}),$$

$$R(xq) = x^{-2}A(x)R(x)S^{-1}, \quad R(x) = \Delta + \cdots \in GL_2(\mathbb{C}\{x^{-1}\})$$

We set

$$M = M(x) = L(x)^{-1}R(x)$$

M is holomorphic on $\mathbb{C}^\times$ and M has simple zeros at $q^{\mathbb{Z}}a_j$.

$$\sigma_q M = x^{-2}RMS^{-1}$$

For any diagonal matrices Γ and Δ,

$$M \sim M' \iff M' = \Gamma^{-1}M\Delta$$

5.1 Precise description of $\mathcal{F}$

We take matrix elements $(m_{11}, m_{12}, m_{21}, m_{22}) \in \mathcal{O}(\mathbb{C}^\times)$ of M.

$$m_{11}m_{22} - m_{12}m_{22} \neq 0.$$

$$(m_{11}m_{22} - m_{12}m_{22})(a_j) = 0.$$

For any $c_1, c_2, d_1, d_2 \in \mathbb{C}^\times$,

$$(m_{11}, m_{12}, m_{21}, m_{22}) \sim (m'_{11}, m'_{12}, m'_{21}, m'_{22})$$

if and only if

$$m'_{jk} = \frac{d_j}{c_i} m_{jk}, \quad j, k = 1, 2.$$

Remark. A **q-analogue of Fuchs' relation**

$$\rho_1 \rho_2 a_1 a_2 a_3 a_4 = \kappa_1 \kappa_2.$$

5.2 Elementary space $V_{k,a}$

For $k \in \mathbb{N}$, $a \in \mathbb{C}^*$,

$$V_{k,a} := \{f \in \mathcal{O}(\mathbb{C}^\times) \,|\, \sigma_q f = a x^{-k} f\}.$$

If $k = 1, 2, 3, \ldots$, $\dim V_{k,a} = k$ and

$$V_{k,a} = \langle \theta_q(-x/\alpha) \,|\, \alpha^k = a \rangle$$

A **natural map**

$$V_{k,a} \times V_{j,b} \to V_{k+j,ab}; \quad (f, g) \mapsto fg$$

Proposition The image of $V_{2,a} \times V_{2,b}$ in $V_{4,ab}$ is a **quadric hypersurface** of equation $XT - YZ = 0$

Proof
Take basis $u, v \in V_{2,a}$, $u', v' \in V_{2,a'}$. Then $(X, Y, Z, T) = (uu', uv', u'v, vv')$ is a basis of $V_{4,ab}$.

5.3 Algebraic description of $\mathcal{F}$: q-$\mathbf{P}_{\mathrm{VI}}$

We consider

$$m_{ij} \in V_{2,\rho_i/\kappa_j}.$$

Then

$$m_{11}m_{22} \in X_1, \quad m_{12}m_{21} \in X_2,$$

where X_1 and X_2 are quadric hypersurfacs in

$$V := V_{4,\rho_1\rho_2/\kappa_1\kappa_2}$$

Since $\det M \neq 0$,

$$(M_1, M_2) = (m_{11}m_{22}, m_{12}m_{21}) \in X_1 \times X_2 \setminus V \times V.$$

The action of diagonal matrices Γ, Δ induces a **scalar action** d_1d_2/c_1c_2.
In the case $x = a_j$ $(j = 1, 2, 3)$ and $k \in \mathbb{Z}$,

$$(m_{11}m_{22} - m_{12}m_{21})(a_j) = 0.$$

Theorem From $\mathcal{F}_{R,S,\underline{a}}$ to

$$\mathcal{G} = \{(f, g) \in X_1 \times X_2 \,|\, (m_{11}m_{22} - m_{12}m_{21})(a_j) = 0 \quad j = 1, 2, 3\}/\mathbb{C}^{\times}$$

is bijective.

10. Other cases

by Jimbo-Sakai, $P(A_3)$, by M. Murata (other)

$$Y(qx,t)=A(x,t)Y(x,t),$$
$$Y(x,qt)=B(x,t)Y(x,t).$$

$$A(x,t)=A_0(t)+xA_1(t)+x^2A_2,$$
$$B(x,t)=f(x,t)(xI+B_0(t)),$$

$$f(x,t)=\begin{cases}\dfrac{x}{(x-a_1qt)(x-a_2qt)} & P(A_3),P(A_4)\\ \dfrac{1}{x-a_1qt} & P(A_5)^\sharp\\ \dfrac{1}{x} & P(A_6)^\sharp\end{cases}$$

The **compatibility condition** leads to q-Painlevé equations

$$A(x,qt)B(x,t)=B(qx,t)A(x,t).$$

$P(A_3)$:

Eigenvalues: $A_0(t) \sim \mathrm{diag}(\theta_1 t, \theta_2 t),\ A_2 = \mathrm{diag}(\kappa_1, \kappa_2)$

$\det A(x,t) = \kappa_1\kappa_2(x - a_1 t)(x - a_2 t)(x - a_3)(x - a_4).$

$P(A_4)$:

Eigenvalues: $A_0(t) \sim \mathrm{diag}(\theta_1 t, \theta_2 t),\ A_2 = \mathrm{diag}(\kappa_1, \mathbf{0})$

$\det A(x,t) = \kappa_1\kappa_2(x - a_1 t)(x - a_2 t)(x - a_3).$

$P(A_5)^\sharp$:

Eigenvalues: $A_0(t) \sim \mathrm{diag}(\theta_1 t, \mathbf{0}),\ A_2 = \mathrm{diag}(\kappa_1, \mathbf{0})$

$\det A(x,t) = \kappa_1\kappa_2 x(x - a_1 t)(x - a_3).$

$P(A_6)^\sharp$:

Eigenvalues: $A_0(t) \sim \mathrm{diag}(\theta_1 t, \mathbf{0}),\ A_2 = \mathrm{diag}(\kappa_1, \mathbf{0})$

$\det A(x,t) = \kappa_1\kappa_2 x^2(x - a_3).$

by M. Murata

10.2 Limit from Painlevé to hypergeometric

$P(A_3)$: $\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)(x-a_4)$.

$\to \det A_1(x,t) = \kappa_1\kappa_2 x^2(x-a_3)(x-a_4)$ Heine

$\to \det A_2(x,t) = \sigma_1\sigma_2(x-a_1)(x-a_2)$ Heine

$P(A_4)$: $\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)$.

$\to \det A_1(x,t) = \kappa_1\kappa_2 x^2(x-a_3)$ q-Kummer

$\to \det A_2(x,t) = \sigma_1\sigma_2(x-a_1)(x-a_2)$ Heine

$P(A_5)^\sharp$: $\det A(x,t) = \kappa_1\kappa_2 x(x-a_1t)(x-a_3)$.

$\to \det A_1(x,t) = \kappa_1\kappa_2 x^2(x-a_3)$ q-Kummer

$\to \det A_2(x,t) = \sigma_1\sigma_2(x-a_1)$ q-Kummer

$P(A_6)^\sharp$: $\det A(x,t) = \kappa_1\kappa_2 x^2(x-a_3)$.

$\to \det A_1(x,t) = \kappa_1\kappa_2 x^2(x-a_3)$ q-Kummer

$\to \det A_2(x,t) = \sigma_1\sigma_2 x^2$ Hahn-Exton

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