Asymptotic analysis of non-abelian Hodge theory in rank 2

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Web-seminar on Painlevé Equations and related topics

Basic notation

Fix the data

- $\mathbb{C}P^1$: complex projective line,
- $r = 2$ rank (i.e., $G = \mathrm{Sl}_2(\mathbb{C})$),
- $p_1, \dots, p_n \in \mathbb{C}P^1$ logarithmic singularities (with local charts z_j),
- $D = p_1 + \cdots + p_n$ parabolic divisor,
- $L = K(D)$ twisted cotangent bundle,
- $\alpha_j^- = \frac{1}{4} < \alpha_j^+ = \frac{3}{4}$ Dolbeault parabolic weights

Hitchin's equations, tame harmonic bundles

We consider Hitchin's equations

$$D^{0,1}\theta = 0$$
$$F_D + [\theta, \theta^{\dagger_h}] = 0$$

for a unitary connection D on a rank 2 smooth Hermitian vector bundle (V, h) and a field $\theta : V \to V \otimes \Omega_C^{1,0}$.
Simpson tame harmonic bundles '90: at the parabolic divisor we require

- θ has first order poles at p_j,
- the eigenvalues of $\mathrm{res}_{p_j}(\theta)$ are equal to 0 (strongly parabolic),
- and with respect to a compatible trivialization,

$$h \approx \mathrm{diag}(|z_j|^{2\alpha_j^-}, |z_j|^{2\alpha_j^+}) = \mathrm{diag}(|z_j|^{\frac{1}{2}}, |z_j|^{\frac{3}{2}})$$

$\rightsquigarrow$ solutions up to gauge equivalence: $\mathcal{M}_{\mathrm{Hod}}$ non-abelian Hodge moduli space, a hyper-Kähler manifold.

de Rham and Dolbeault structures

Two Kähler structures on $\mathcal{M}_{\text{Hod}}$ have a geometric meaning:

- de Rham: $\mathcal{M}_{\text{dR}}$ parameterising certain poly-stable parabolic connections (E, ∇) with regular singularities
- Dolbeault: $\mathcal{M}_{\text{Dol}}$ parameterising certain poly-stable parabolic Higgs bundles $(\mathcal{E}, \theta)$ with first-order poles.

By non-abelian Hodge theory, $\mathcal{M}_{\text{dR}}$ and $\mathcal{M}_{\text{Dol}}$ are diffeomorphic to each other (via $\mathcal{M}_{\text{Hod}}$):

$$\text{NAHT}\colon \mathcal{M}_{\text{Dol}} \xrightarrow{\sim} \mathcal{M}_{\text{dR}}.$$

Character variety, Riemann–Hilbert correspondence

Character variety $\mathcal{M}_{\mathrm{B}}$: moduli space parameterising (filtered) local systems ρ on $\mathbb{C}P^1 \setminus D$, with eigenvalues on a simple positive loop around p_j given by

$$c_j^{\pm} = \exp(-2\pi\sqrt{-1}\alpha_j^{\pm}) = \pm\sqrt{-1}.$$

Regular-singular Riemann–Hilbert correspondence: bi-analytic map

$$\mathrm{RH}\colon \mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}}.$$

Conclusion: $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ are all diffeomorphic to each other (and to $\mathcal{M}_{\mathrm{Hod}}$), in particular

$$(\mathrm{RH} \circ \mathrm{NAHT})^*\colon H^{\bullet}(\mathcal{M}_{\mathrm{B}}, \mathbb{Q}) \xrightarrow{\cong} H^{\bullet}(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}).$$

Perverse filtration on Dolbeault spaces

Hitchin '87: for $\mathcal{M}_{\mathrm{Dol}}$ a Dolbeault moduli space there exists a surjective proper algebraic map of quasi-projective varieties

$$H\colon \mathcal{M}_{\mathrm{Dol}} \to Y = \mathbb{C}^N.$$

This endows $H^\bullet(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$ with a perverse filtration P defined by

$$P^p\mathrm{H}^*(Y, \mathrm{R}H_*\underline{\mathbb{Q}}_{\mathcal{M}}) = \mathrm{Im}(\mathrm{H}^*(Y, {}^{\mathfrak{p}}\tau_{\leq -p}\mathrm{R}H_*\underline{\mathbb{Q}}_{\mathcal{M}}) \to \mathrm{H}^*(Y, \mathrm{R}H_*\underline{\mathbb{Q}}_{\mathcal{M}})),$$

where

$$^{\mathfrak{p}}\tau_{\leq i}\colon D^b_{\mathrm{constr}}(Y, \mathbb{Q}) \to {}^{\mathfrak{p}}D^{\leq i}_{\mathrm{constr}}(Y, \mathbb{Q})$$

are the Beilinson–Bernstein–Deligne truncation functors.

Weight filtration on Betti spaces

$\mathcal{M}_{\mathrm{B}}$ is an affine algebraic variety. Deligne's Hodge II. ('71) $\Rightarrow H^*(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ carries a Mixed Hodge Structure, in particular a weight filtration W.

Known: there exists a spectral sequence depending on the cohomology groups of a smooth compactification $\widetilde{\mathcal{M}}_{\mathrm{B}}$ of $\mathcal{M}_{\mathrm{B}}$ and the combinarorics of a compactifying divisor, abutting to W.

Hausel–Rodriguez-Villegas '08: for character varieties one has $W_{2k} = W_{2k-1}$.

$P = W$ conjecture

Theorem (de Cataldo–Hausel–Migliorini '12)

If C is a smooth projective curve and $r = 2$, then for the Dolbeault and Betti spaces corresponding to each other under non-abelian Hodge theory and the Riemann–Hilbert correspondence, the filtrations P and W get mapped into each other:

$$(\mathrm{RH} \circ \mathrm{NAHT})^* W_{2i} H^k(\mathcal{M}_{\mathrm{B}}, \mathbb{Q}) = P^i H^k(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q}).$$

Conjecture (de Cataldo–Hausel–Migliorini '12)

The same property holds for any rank r.

Progress on $P = W$ conjecture

- M. de Cataldo, D. Maulik and J. Shen '19 established it for curves of genus 2.
- J. Shen and Z. Zhang '18 proved it for five infinite families of moduli spaces of parabolic Higgs bundles over $\mathbb{C}P^1$.
- C. Felisetti and M. Mauri '20 proved it for character varieties admitting a symplectic resolution, i.e. in genus 1 and arbitrary rank and genus 2 and rank 2.
- Sz '19 established it for complex 2-dimensional moduli spaces of rank 2 Higgs bundles with irregular singularities over $\mathbb{C}P^1$ corresponding to the Painlevé cases.
- L. Katzarkov, A. Harder and V. Przyjalkowski '19 have formulated a version for log-Calabi–Yau manifolds and their mirror pairs.
- ...

Geometric $P = W$ conjecture

- L. Katzarkov, A. Noll, P. Pandit and C. Simpson, 2015: conjectured a certain homotopy commutativity property for $\mathrm{RH} \circ \mathrm{NAHT}$ (see next Section).
- Trivial consequence of this homotopy commutativity: $|\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma})|$ has homotopy type S^{N-1}, where $2N = \dim_{\mathbb{C}} \mathcal{M}_{\mathrm{B}}$.
- A. Komyo, 2015: proved that for $n = 5$ with our notations, $|\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma})|$ is homotopy equivalent to S^3.
- C. Simpson, 2015: generalized the homotopy equivalence assertion to any $n \geq 5$, and named the homotopy commutativity assertion "Geometric P = W conjecture".
- M. Mauri, E. Mazzon and M. Stevenson, 2018: showed that $|\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}|$ for the $\mathrm{Gl}(n, \mathbb{C})$ character variety of a 2-torus is homeomorphic to S^{2n-1}.

Map to Danilov complex

Let $\widetilde{\mathcal{M}}_{\mathrm{B}}$ be a smooth compactification of $\mathcal{M}_{\mathrm{B}}$ by a simple normal crossing divisor D and denote by $\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c},\vec{\gamma})$ the nerve (aka. dual) simplicial complex of D:

- to each irreducible component $D_i \subset D \rightsquigarrow$ a 0-cell in $\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c},\vec{\gamma})_0$,
- to each nonempty intersection $D_i \cap D_{i'} \neq \varnothing \rightsquigarrow$ a 1-cell in $\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c},\vec{\gamma})_1$,
- etc.

Let $T_i \subset \mathcal{M}_{\mathrm{B}}$ be a punctured tubular neighbourhood of D_i and

$$T = \cup_i T_i.$$

There exists (up to homotopy) a natural map

$$\Phi\colon T \to |\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c},\vec{\gamma})|.$$

Geometric $P = W$ conjecture in the Painlevé 6 case

Let $n = 4$, $D = 0 + 1 + t + \infty$.

Theorem (Sz '19 (to appear in Adv. Math.), Némethi–Sz '20 (IMRN))

For some sufficiently large compact $K \subset \mathcal{M}_{\mathrm{B}}$ there exists a homotopy commutative square

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}} \setminus \psi^{-1}(K) & \xrightarrow{\psi} & \mathcal{M}_{\mathrm{B}} \setminus K \\ \Big\downarrow H & & \Big\downarrow \Phi \\ D^{\times} & \longrightarrow & |\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma})|. \end{array}$$

Here, $D^{\times} = \mathbb{C} - B_R(0) \subset Y$ *and* $\psi = \mathrm{RH} \circ \mathrm{NAHT}$.

Hitchin base and standard spectral curve

Proof of the Theorem based on Sz. 1906.01856. Assumptions $\Rightarrow \mathrm{tr}(\theta) \equiv 0$, Hitchin base:

$$H^0(\mathbb{C}P^1, K^2(0 + 1 + t + \infty)) \cong \mathbb{C},$$

spanned by

$$\frac{(\mathrm{d}z)^{\otimes 2}}{z(z-1)(z-t)}.$$

Set $L = K(0 + 1 + t + \infty)$, and take the canonical section

$$\zeta \frac{\mathrm{d}z}{z(z-1)(z-t)}$$

of $p_L^* L$ over $\mathrm{Tot}(L)$. In $\mathrm{Tot}(L)$ we consider the curve

$$\tilde{X}_{1,0} = \{(z, \zeta) : \quad \zeta^2 + z(z-1)(z-t) = 0\}.$$

Rescaling of spectral curve

For $R \gg 0, \varphi \in \mathbb{R}/2\pi\mathbb{Z}$ let $(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$ be a rank 2 logarithmic Higgs bundle over $\mathbb{C}P^1$ with

$$\det(\theta_{R,\varphi}) = -Re^{\sqrt{-1}\varphi} \in H^0(\mathbb{C}P^1, K^2(0+1+t+\infty)).$$

Its spectral curve is

$$\tilde{X}_{R,\varphi} = \left\{(z,\zeta) : \quad \det\left(\theta_{R,\varphi} - \zeta\frac{\mathrm{d}z}{z(z-1)(z-t)}\right) = 0\right\} \subset \mathrm{Tot}(L),$$

with natural projection given by

$$\begin{aligned} p : \tilde{X}_{R,\varphi} &\to \mathbb{C}P^1 \\ (z,\zeta) &\mapsto z. \end{aligned}$$

We have

$$(z,\zeta) \in \tilde{X}_{R,\varphi} \Leftrightarrow (z, \sqrt{-1}R^{-\frac{1}{2}}e^{-\sqrt{-1}\varphi/2}\zeta) \in \tilde{X}_{1,0}.$$

Abelianization

Set

$$\omega = \frac{\mathrm{d}z}{\sqrt{z(z-1)(z-t)}}.$$

T. Mochizuki (2016): on simply connected open sets $U \subset \mathbb{C} \setminus \{0, 1, t\}$ there is a gauge $e_1(z), e_2(z)$ of $\mathcal{E}$ with respect to which

$$\theta_{R,\varphi}(z) - \begin{pmatrix} \sqrt{R}e^{\sqrt{-1}\varphi/2} & 0 \\ 0 & -\sqrt{R}e^{\sqrt{-1}\varphi/2} \end{pmatrix} \omega \to 0$$

as $R \to \infty$, and the Hermitian–Einstein metric h is close to an abelian model h_{ab}.

Crucial observation

Since ω has ramification at $0, 1, t, \infty$, along a simple loop γ around these points, the local sections $e_1(z), e_2(z)$ get interchanged.

Non-abelian Hodge theory at large R

The connection matrix associated to $(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$ is

$$a_{R,\varphi}(z,\bar{z}) = \theta_{R,\varphi}(z) + \overline{\theta_{R,\varphi}(z)} + b_{R,\varphi}$$
$$\approx \sqrt{R}\begin{pmatrix} e^{\sqrt{-1}\varphi/2}\omega + e^{-\sqrt{-1}\varphi/2}\bar{\omega} & 0 \\ 0 & -e^{\sqrt{-1}\varphi/2}\omega - e^{-\sqrt{-1}\varphi/2}\bar{\omega} \end{pmatrix}$$
$$+ b_{R,\varphi}$$

where $d + b_{R,\varphi}$ is the Chern connection associated to the holomorphic structure of $\mathcal{E}$ and h_{ab}. So $b_{R,\varphi}$ takes values in $\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1))$.

Monodromy matrices at large R

The monodromy matrices of the connection $\mathrm{d} + a_{R,\varphi}$ along a simple loop γ_j around $j \in \{0, 1, t\}$ are

$$B_j(R,\varphi) = T \exp \oint_{\gamma_j} -a_{R,\varphi}(z,\bar{z}) = TA_j(R,\varphi) \cdot$$

$$\exp\sqrt{R}\begin{pmatrix} -e^{\sqrt{-1}\varphi/2}\pi_j - e^{-\sqrt{-1}\varphi/2}\overline{\pi_j} & 0 \\ 0 & e^{\sqrt{-1}\varphi/2}\pi_j + e^{-\sqrt{-1}\varphi/2}\overline{\pi_j} \end{pmatrix}$$

where we have set

$$\pi_j = \oint_{\gamma_j} \omega, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $A_j(R,\varphi) \in \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1))$ is the monodromy of the Chern connection.

Products of monodromy matrices at large R

Setting

$$A_j(R,\varphi) = \begin{pmatrix} e^{\sqrt{-1}\mu_j} & 0 \\ 0 & e^{-\sqrt{-1}\mu_j} \end{pmatrix}$$

and

$$d_{01}(R,\varphi) = \exp\left(\sqrt{-1}(\mu_1 - \mu_0) + 2\sqrt{R}\Re(e^{\sqrt{-1}\varphi/2}(\pi_0 - \pi_1))\right)$$

it follows that

$$B_0(R,\varphi)B_1(R,\varphi) \approx \begin{pmatrix} d_{01}(R,\varphi) & 0 \\ 0 & d_{01}(R,\varphi)^{-1} \end{pmatrix}.$$

Affine coordinates on the Betti space

Let us set

$$\begin{aligned} x_1(R,\varphi) &= \mathrm{tr}(B_0(R,\varphi)B_1(R,\varphi)) \\ x_2(R,\varphi) &= \mathrm{tr}(B_t(R,\varphi)B_0(R,\varphi)) \\ x_3(R,\varphi) &= \mathrm{tr}(B_1(R,\varphi)B_t(R,\varphi)). \end{aligned}$$

These co-ordinates satisfy Fricke–Klein cubic relation:

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - s_1x_1 - s_2x_2 - s_3x_3 + s_4 = 0$$

for some $s_1, s_2, s_3, s_4 \in \mathbb{C}$. Compactifying divisor of $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$:

$$\begin{aligned} D &= (x_1x_2x_3) \subset \mathbb{C}P^2_\infty \\ &= L_1 \cup L_2 \cup L_3 \end{aligned}$$

where L_i are lines pairwise intersecting each other in points P_1, P_2, P_3.

Dual boundary complex

The nerve complex $\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma})$ of D has vertices v_1, v_2, v_3 corresponding to line components

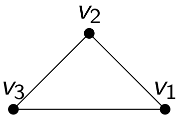
$$L_1 = [0 : 0 : x_2 : x_3], \quad L_2 = [0 : x_1 : 0 : x_3], \quad L_3 = [0 : x_1 : x_2 : 0]$$

of D and edges

$$[v_1 v_2], \quad [v_2 v_3], \quad [v_3 v_1]$$

corresponding to intersection points of the components:

$$[0 : 0 : 0 : 1], \quad [0 : 1 : 0 : 0], \quad [0 : 0 : 1 : 0]$$



Simpson's map

Let T_i be an open tubular neighbourhood of L_i in $\widetilde{\mathcal{M}}_{\mathrm{B}}$ and set

$$T = T_1 \cup T_2 \cup T_3.$$

Let $\{\phi_i\}$ be a partition of unity subordinate to the cover of T by $\{T_i\}$. Define the map

$$\begin{aligned} \Phi : T &\to \mathbb{R}^3 \\ x &\mapsto \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix}. \end{aligned}$$

Then,

$$\mathrm{Im}(\Phi) = [v_1 v_2] \cup [v_2 v_3] \cup [v_3 v_1] \cong S^1.$$

Asymptotic of Riemann–Hilbert correspondence at large R

Fix $R \gg 0$ and let $\varphi \in [0, 2\pi)$ vary. Need to show: the loop

$$\Phi \circ \mathrm{RH}(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi})$$

generates $\pi_1(\mathrm{Im}(\Phi)) \cong \mathbb{Z}$.
Key fact: for $d \in \mathbb{C}$ with $|\Re(d)| \gg 0$ we have

$$|2\cosh(d)| \approx e^{|d|}.$$

This implies

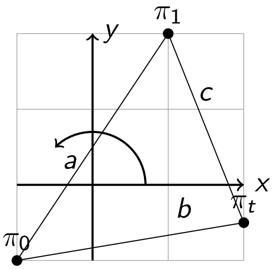
$$|x_1(R,\varphi)| \approx \exp\left(2\sqrt{R}|\Re(e^{\sqrt{-1}\varphi/2}(\pi_0 - \pi_1))|\right),$$
$$|x_2(R,\varphi)| \approx \exp\left(2\sqrt{R}|\Re(e^{\sqrt{-1}\varphi/2}(\pi_t - \pi_0))|\right),$$
$$|x_3(R,\varphi)| \approx \exp\left(2\sqrt{R}|\Re(e^{\sqrt{-1}\varphi/2}(\pi_1 - \pi_t))|\right).$$

Rotating triangle

Let $\Delta \subset \mathbb{C}$ be the triangle with vertices π_0, π_1, π_t, assume Δ is non-degenerate. Denote its sides by

$$a = \pi_0 - \pi_1, \quad b = \pi_t - \pi_0, \quad c = \pi_1 - \pi_t.$$

Let us denote by $e^{\sqrt{-1}\varphi/2}\Delta$ the triangle obtained by rotating Δ by angle $\varphi/2$ in the positive direction, with sides $e^{\sqrt{-1}\varphi/2}a$, $e^{\sqrt{-1}\varphi/2}b$, $e^{\sqrt{-1}\varphi/2}c$.



Critical angles

Lemma

For each side a, b, c there exists exactly one value $\varphi_a, \varphi_b, \varphi_c \in [0, 2\pi)$ such that $e^{\sqrt{-1}\varphi_a/2} a$ (respectively $e^{\sqrt{-1}\varphi_b/2} b$, $e^{\sqrt{-1}\varphi_c/2} c$) is purely imaginary. The function

$$\Re(e^{\sqrt{-1}\varphi/2} b) - \Re(e^{\sqrt{-1}\varphi/2} c)$$

changes sign at $\varphi = \varphi_a$. Similar statements hold with a, b, c permuted.

Definition

$\varphi_a, \varphi_b, \varphi_c$ are the critical angles associated to the sides a, b, c respectively.

Arc decomposition of the circle

The critical angles decompose S^1 into three closed arcs

$$S^1 = I_1 \cup I_2 \cup I_3$$

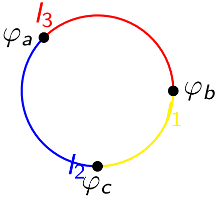
satisfying:

$$\max(|\Re(e^{\sqrt{-1}\varphi/2}(\pi_0-\pi_1))|, |\Re(e^{\sqrt{-1}\varphi/2}(\pi_t-\pi_0))|, |\Re(e^{\sqrt{-1}\varphi/2}(\pi_1-\pi_t))|)$$

is attained

- by $|\Re(e^{\sqrt{-1}\varphi/2}(\pi_0 - \pi_1))|$ for $\varphi \in I_1$,
- by $|\Re(e^{\sqrt{-1}\varphi/2}(\pi_t - \pi_0))|$ for $\varphi \in I_2$,
- and by $|\Re(e^{\sqrt{-1}\varphi/2}(\pi_1 - \pi_t))|$ for $\varphi \in I_3$.

Arc decomposition of the circle



Limiting Riemann–Hilbert map

We deduce

- for $\varphi \in \mathrm{Int}(I_1)$, we have
$$[x_0 : x_1 : x_2 : x_3] \to [0 : 1 : 0 : 0],$$
- for $\varphi \in \mathrm{Int}(I_2)$, we have
$$[x_0 : x_1 : x_2 : x_3] \to [0 : 0 : 1 : 0],$$
- for $\varphi \in \mathrm{Int}(I_3)$, we have
$$[x_0 : x_1 : x_2 : x_3] \to [0 : 0 : 0 : 1].$$

Limiting Simpson's map

Applying Simpson's map Φ to the previous limits we get that

- for $\varphi \in \mathrm{Int}(I_1)$, we have

$$\Phi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in [v_2 v_3],$$

- for $\varphi \in \mathrm{Int}(I_2)$, we have

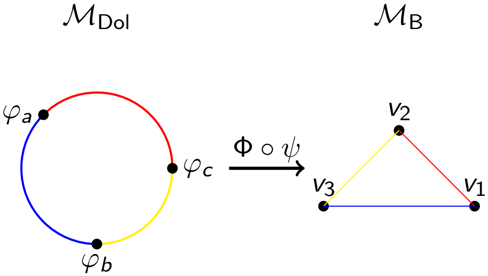
$$\Phi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in [v_3 v_1],$$

- for $\varphi \in \mathrm{Int}(I_3)$, we have

$$\Phi(\mathcal{E}_{R,\varphi}, \theta_{R,\varphi}) \in [v_1 v_2].$$

Thus, Φ sends a generator of $\pi_1(S^1_\varphi)$ into a generator of $\pi_1(\mathrm{Im}(\Phi))$.

Limiting composed map



Result in Garnier case with 5 logarithmic points

From now on we let $n = 5$, based on arXiv:2103.00932.

Theorem

There exists a value of the map

$$\Phi \circ \mathrm{RH} \circ \psi \circ \sigma \colon S^3 \to S^3$$

whose preimages lie in a tubular neighbourhood of a curve $C \subset S^3$, and the derivative of the map at one of the preimages is invertible.

Remark

For the Geometric $P = W$ conjecture, we would also need that the given value is only attained at one point of S^3. We strongly believe this is true, however have no rigorous proof as of now.

Hitchin base and map

Let $(\mathcal{E}, \theta)$ be a strongly parabolic Higgs bundle of rank 2 with 5 logarithmic points.
Again, we have $\mathrm{tr}(\theta) \equiv 0$. Hitchin base:

$$\mathcal{B} = \{q : \quad q(t_j) = 0 \text{ for all } 0 \leq j \leq 4\} \subset H^0(\mathbb{C}P^1, L^{\otimes 2}) \cong \mathbb{C}^7,$$

so $\dim_{\mathbb{C}}(\mathcal{B}) = 2$. Hitchin map:

$$H \colon \mathcal{M}_{\mathrm{Dol}}(\vec{0}, \vec{\alpha}) \to \mathcal{B}$$
$$(\mathcal{E}, \theta) \mapsto -\det(\theta)$$

Spectral curve

For $q \in S^3_1 \subset \mathcal{B}$ we write $\zeta_\pm(Rq, z)$ for the roots of

$$\zeta^2 - Rq = 0,$$

specifically

$$\zeta_\pm(Rq, z) = \pm\sqrt{Rq(z, 1)}.$$

Denote the corresponding meromorphic 1-forms by

$$Z_\pm(Rq, z) = \pm\sqrt{Rq(z, 1)} \frac{\mathrm{d}z}{\prod_{j=0}^{4}(z - t_j)}.$$

We denote by

$$X_{Rq} = \{([z : w], \pm\sqrt{Rq(z, w)})\} \to \mathbb{C}P^1$$

the Riemann surface of the bivalued function $\zeta_\pm(Rq, z)$.

Ramification divisor and Hopf fibration

We set

$$\Delta_q = \{z \in \mathbb{C} : \quad q(z) = 0\}$$

for the ramification divisor of X_{Rq}. We then have

$$\Delta_q = \{t_0, t_1, t_2, t_3, t_4, t(q)\}$$

for some $t(q) \in \mathbb{C}P^1$. Namely,

$$q(z) = \frac{(az - b)\mathrm{d}z^{\otimes 2}}{\prod_{j=0}^{4}(z - t_j)}.$$

for some $(a, b) \in \mathbb{C}^2$. The map

$$\begin{aligned} t \colon S_1^3 &\to \mathbb{C}P^1 \\ q &\mapsto t(q) \end{aligned}$$

is the Hopf fibration.

Idea of proof

We fix a generic element $q \in S_1^3$ and consider $(\mathcal{E}, \theta) \in \mathcal{M}_{\text{Dol}}(\vec{0}, \vec{\alpha})$ such that

$$H(\mathcal{E}, \theta) = q.$$

For $R > 0$ we have

$$H(\mathcal{E}, \sqrt{R}\theta) = Rq.$$

It is then possible to express the $R \to \infty$ asymptotic behaviour of $\Phi \circ \psi$ in function of $\int_\gamma Z_\pm(q, z)$ over various paths γ in $\mathbb{C}P^1$, up to factors belonging to $U(1)$.
We choose a smooth section

$$\sigma \colon S_1^3 \to \mathcal{M}_{\text{Dol}}(\vec{0}, \vec{\alpha})$$

to get rid of the $U(1)$ factors.

Asymptotic abelianization

Let $h_{\sqrt{R}}$ and $\nabla_{\sqrt{R}}$ denote the Hermite–Einstein metric and integrable connection associated to $(\mathcal{E}, \sqrt{R}\theta)$. Introduce

$$\nabla^{\text{model}}_{\sqrt{R}} = \nabla_{h_{q,\infty}} + \begin{pmatrix} 2\Re Z_+(Rq, z) & 0 \\ 0 & 2\Re Z_-(Rq, z) \end{pmatrix}.$$

where $h_{q,\infty}$ is some explicit abelian solution of Hitchin's equation (i.e., with values in $S(U(1) \times U(1))$) and $\nabla_{h_{q,\infty}}$ the corresponding unitary connection.

Theorem (T. Mochizuki '16)

Over any simply connected compact set $K \subset \mathbb{C} \setminus \Delta_q$ there exists a gauge transformation $g_{\sqrt{R}}$ such that

$$g_{\sqrt{R}} \cdot \nabla_{\sqrt{R}} - \nabla^{\text{model}}_{\sqrt{R}} \to 0$$

(measured with respect to $h_{\sqrt{R}}$) as $R \to \infty$, uniformly over K.

Fiducial solution, Painlevé 3

R. Mazzeo, J. Swoboda, H. Weiss, F. Witt '16 (near ramification points $t(q)$), L. Fredrickson, R. Mazzeo, J. Swoboda, H. Weiss '20 (near parabolic points D): local models for the $R \gg 0$ behaviour of $h_{\sqrt{R}}$ and $\nabla_{\sqrt{R}}$, called fiducial solutions.
Near $t(q)$: let $\ell_{\sqrt{R}}$ be the solution of the Painlevé 3-type equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\right)\ell_{\sqrt{R}} = 8R\tilde{r}\sinh(2\ell_{\sqrt{R}})$$

satisfying the boundary behaviours

$$\ell_{\sqrt{R}}(\tilde{r}) \approx -\frac{1}{2}\log(\tilde{r}), \quad \tilde{r} \to 0+$$

$$\ell_{\sqrt{R}}(\tilde{r}) \approx \frac{1}{\pi}K_0\left(\frac{8}{3}\sqrt{R\tilde{r}^3}\right) \approx \frac{\sqrt{3}}{2\pi\sqrt{2}\sqrt[4]{R\tilde{r}^3}}e^{-\frac{8}{3}\sqrt{R\tilde{r}^3}}, \quad \tilde{r} \to \infty,$$

with K_0 the modified Bessel function of order 0.

Fiducial solution, approximate solution

Then, for a co-ordinate $\tilde{z}$ on the disc $|\tilde{z}| < 1$ introduce a unitary connection and Higgs field:

$$A^{\mathrm{fid}}_{\sqrt{R}} = \left(\frac{1}{8} + \frac{1}{4}\tilde{r}\partial_{\tilde{r}}\ell_{\sqrt{R}}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 2\sqrt{-1}\mathrm{d}\tilde{\varphi}$$

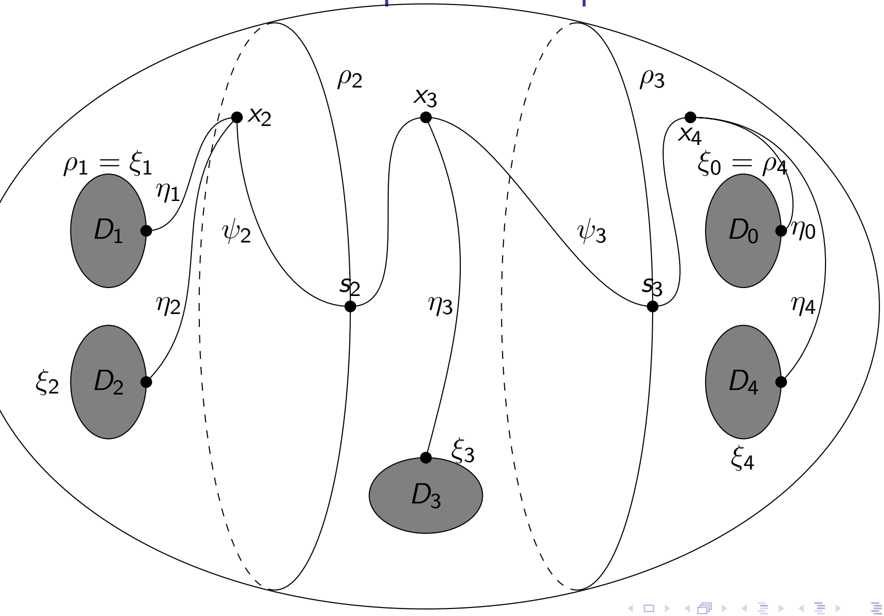
$$\theta^{\mathrm{fid}}_{\sqrt{R}} = \begin{pmatrix} 0 & \tilde{r}^{1/2}e^{\ell_{\sqrt{R}}(\tilde{r})} \\ \tilde{z}\tilde{r}^{-1/2}e^{-\ell_{\sqrt{R}}(\tilde{r})} & 0 \end{pmatrix}\mathrm{d}\tilde{z}.$$

Gluing construction of the fiducial solution and Mochizuki's abelian form $\rightsquigarrow$ approximate solution $h^{\mathrm{appr}}_{\sqrt{R}}$.

Theorem (MSWW '16, FMSW '20)

Assume that all the zeroes of q are simple. Then, there exists a small perturbation (for an appropriate Hölder norm) of the Hermitian metric $h^{\mathrm{appr}}_{\sqrt{R}}$ that satisfies Hitchin's equation for $(\mathcal{E}, \sqrt{R}\theta)$.

Pair-of-pants decomposition



Simpson's Fenchel–Nielsen co-ordinates

Simpson '16: $\mathcal{M}_{\mathrm{B}}$ carries complex length co-ordinates

$$t_i = \operatorname{tr} \mathrm{RH}(\nabla)[\rho_i] \in \mathbb{C} \quad (i \in \{2, 3\}),$$

and complex twist co-ordinates

$$[p_i : q_i] \in \mathbb{C}P^1 \quad (i \in \{2, 3\}),$$

subject to the condition

$$p_i^2 + t_i p_i q_i + q_i^2 \neq 0.$$

Boundary divisor of character variety

Introduce

$\mathrm{Q} = \{(t, [p:q]) \in (\mathbb{C} \setminus \{\pm 2\}) \times \mathbb{C}P^1$ satisfying $p^2 + tpq + q^2 \neq 0\}$.

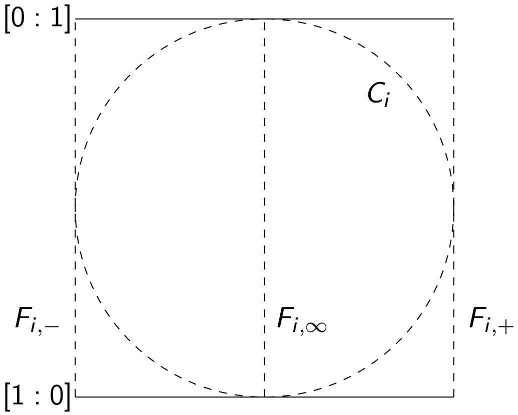
Simpson: homotopy type of the dual boundary complex of $\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma})$ agrees with the one of Q^2

$$\mathbb{D}\partial\mathcal{M}_{\mathrm{B}}(\vec{c}, \vec{\gamma}) \sim \mathbb{D}\partial\mathrm{Q}^2 \sim \mathbb{D}\partial\mathrm{Q} * \mathbb{D}\partial\mathrm{Q} \sim S^1 * S^1 \sim S^3.$$

Boundary divisor of Q

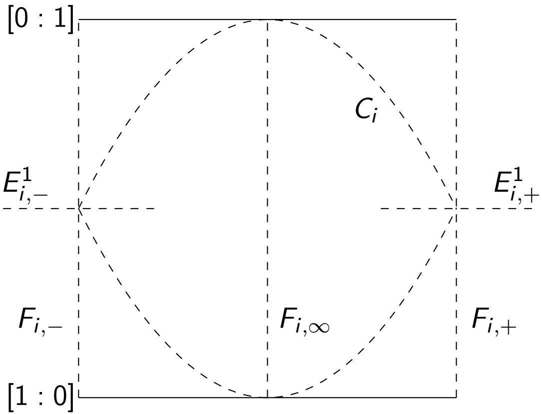
Set

$$C_i = (p_i^2 + t_i p_i q_i + q_i^2), \quad F_{i,\pm} = \{t = \pm 2\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1.$$



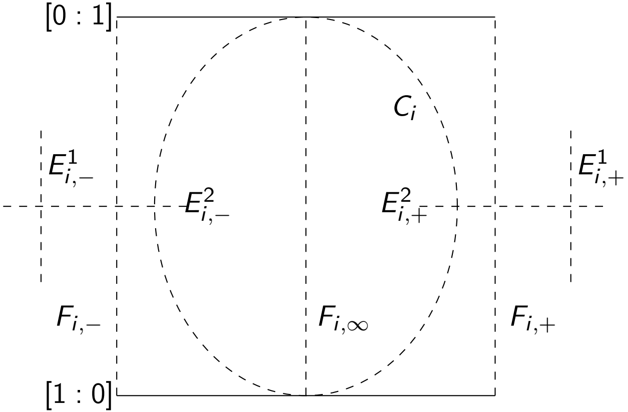
This is not simple normal crossing $\Rightarrow$ one needs to apply blow-ups.

First blow-up



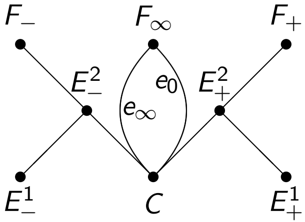
Still not SNC.

Second blow-up



This is SNC.

Dual complex of ∂Q



Parallel transport map

For any loop γ in $\mathbb{C}P^1 \setminus \Delta_q$ let us write

$$\mathrm{RH}(\nabla_{\sqrt{R}})[\gamma] = \begin{pmatrix} a(\gamma, q, R) & b(\gamma, q, R) \\ c(\gamma, q, R) & d(\gamma, q, R) \end{pmatrix}.$$

For $0 \leq j \leq 2$ introduce

$$\pi_j(q) = \int_{x_2}^{t_j} Z_+(q, z) \in \mathbb{C},$$

$$\tau_j(q) = \frac{a t_j - b}{\prod_{0 \leq k \leq 4, k \neq j} (t_j - t_k)} \in \mathbb{C}.$$

Asymptotics of complex length co-ordinates t_2

Proposition

Fix $q \in S_1^3$ and consider the loop $\gamma = \rho_2$. In case $\Re(\pi_1 - \pi_2) \neq 0$ there exists a complex 1-parameter family of sections σ of the Hitchin map H such that as $R \to \infty$ we have the limit

$$t_2(q, R) = \exp\left(4\sqrt{R}|\Re(\pi_1 - \pi_2)|\right) + o(1).$$

In case $\Re(\pi_1 - \pi_2) = 0$ the limit of $t_2(q, R)$ as $R \to \infty$ exists and is finite.

Proposition

Fix $q \in S_1^3$ and assume $\pi_1(q) \neq \pi_2(q)$. Then there exists a unique $\varphi_2 \in [0, 2\pi)$ such that $t_2(e^{\sqrt{-1}\varphi_2} q, R)$ is bounded as $R \to \infty$.

Asymptotics of complex length co-ordinates

Proposition

Let $q \in S_1^3$ satisfy

$$\Re(\pi_4(q) - \pi_0(q)) \neq 0 \neq \Re(\pi_1(q) - \pi_2(q)).$$

Then there exists a section σ of H such that we have limits

$$\lim_{R\to\infty} t_2(q,R) \exp\left(-4\sqrt{R}|\Re(\pi_1(q) - \pi_2(q))|\right) = 1$$

and

$$\lim_{R\to\infty} t_3(q,R) \exp\left(-4\sqrt{R}|\Re(\pi_4(q) - \pi_0(q))|\right) = 1.$$

Limit of complex twist co-ordinate $[p_2 : q_2]$

Proposition

Fix $q \in S_1^3$ such that $\Re(\pi_2 - \pi_1) \neq 0$. Then, the complex twist co-ordinate $[p_2 : q_2]$ associated to Rq converges to $[0 : 1]$ as $R \to \infty$ if the conditions

$$\int_{\psi_2} \Re Z_+ < 2\Re(2\sqrt{s_2\tau_2} - \sqrt{s_2\tau_1} - \sqrt{s_3\tau_3})$$
$$|\Re(\pi_1 - \pi_2)| = 2\sqrt{s_2}\Re(\sqrt{\tau_1} - \sqrt{\tau_2})$$

hold for one choice of a square root Z_+ of Q.
On the other hand, under the condition

$$|\Re(\pi_1 - \pi_2)| \neq 2\sqrt{s_2}\Re(\sqrt{\tau_1} - \sqrt{\tau_2})$$

$[p_2 : q_2]$ converges to $[1 : 0]$.

Asymptotics of complex twist co-ordinate $[p_2 : q_2]$

Specifically, in the first situation we have

$$\frac{p_2}{q_2} \approx \exp 4\sqrt{R}\Re\left(\int_{\psi_2} Z_+(q) - 2(2\sqrt{s_2\tau_2}(q) - \sqrt{s_2\tau_1}(q) - \sqrt{s_3\tau_3}(q))\right).$$

This behaviour follows from some miraculous cancellations.
Conclusion:

- the behaviour $\frac{p_2}{q_2} \to \infty$ is generic,
- the challenge is to find $q \in S_1^3$ such that $\frac{p_2}{q_2} \to 0$.

Geometry of period integrals

Define the open subset

$$U_2(s_2) \subset S_1^3$$

by the conditions

$$0 \neq \pi_1(q) - \pi_2(q) \neq \pm 2\sqrt{s_2}(\sqrt{\tau_1}(q) - \sqrt{\tau_2}(q)).$$

For every $q \in U_2$ there exists a unique $\varphi^* \in [0, 2\pi)$ such that

$$\Re(\pi_1(e^{\sqrt{-1}\varphi^*}q) - \pi_2(e^{\sqrt{-1}\varphi^*}q)) = 2\sqrt{s_2}\Re(\sqrt{\tau_1}(e^{\sqrt{-1}\varphi^*}q) - \sqrt{\tau_2}(e^{\sqrt{-1}\varphi^*}q$$

This provides a smooth section of the Hopf fibration

$$S_2\colon t(U_2) \to S_1^3$$
$$[a : b] \mapsto e^{\sqrt{-1}\varphi^*(q)}q$$

Finding small $[p_2 : q_2]$

We make the choices

$$t_0 = -\frac{1}{k}, \quad t_1 = 0, \quad t_2 = 1, \quad t_3 = -1, \quad t_4 = \frac{1}{k}$$

for some $0 < k < 1$.

Proposition

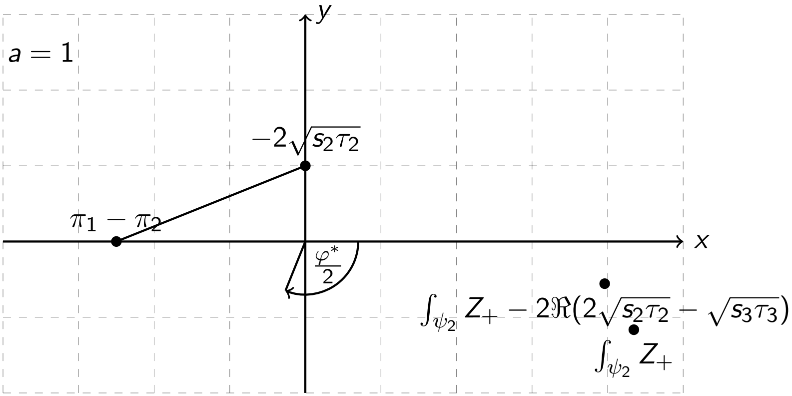
Let $q = S_2(t_1)$. Then q belongs to $U_2(s_2)$ for every $s_2 > 0$, and we have $\Re(\pi_1(q) - \pi_2(q)) \neq 0$. Moreover, there exist distinct points $x_2, x_3 \in \mathbb{C}P^1 \setminus D$ and

$$\rho = \rho(q, t_0, \ldots, t_4, x_2, x_3) > 0$$

such that for every $0 < s_2, s_3 < \rho$ we have $[p_2 : q_2] \to [0 : 1]$ as $R \to \infty$.

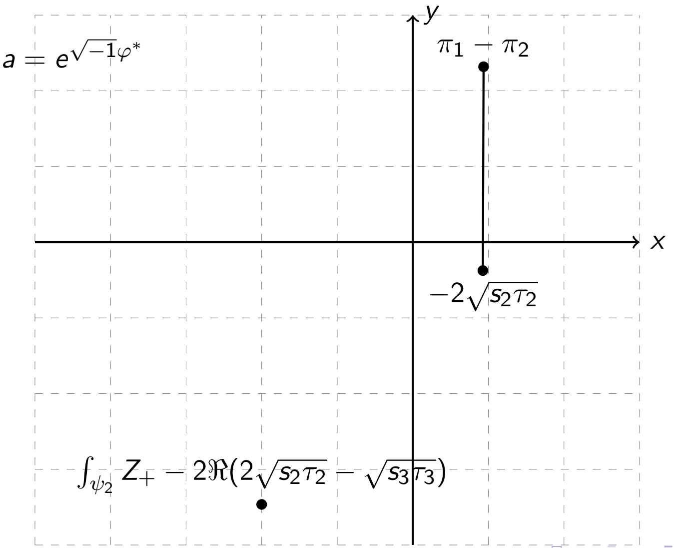
Idea of proof to find small $[p_2 : q_2]$

Rotating triangles, again. Before:



Idea of proof to find small $[p_2 : q_2]$

After:



Finding small $[p_2 : q_2]$ and $[p_3 : q_3]$ simultaneously

Proposition

There exist $0 < s_2, s_3, s_4 < \rho''$ such that $S_2(t_1) = S_3(t_1)$. For the choice $q = S_2(t_1)$, we have $[p_2 : q_2] \to [0 : 1]$ and $[p_3 : q_3] \to [0 : 1]$ as $R \to \infty$.

I suspect that the value

$$\Phi \circ \mathrm{RH} \circ \psi \circ \sigma(RS_2(t_1))$$

is regular with a single preimage. I can show that the derivative at $RS_2(t_1)$ is of full rank.
All its preimages lie in a tubular neighbourhood of the curve

$$C = \mathrm{Im}(S_2) \cap \mathrm{Im}(S_3).$$

Needs to be done: it admits a unique preimage.