# Deformation of Moduli Spaces of Meromorphic Connections on the Riemann Sphere via Unfolding of Irregular Singularities

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# WHAT DO WE WANT?

- Good holomorphic families of ODE on $\mathbb{P}^1$ arising from confluence of thier singular points.

E.g.: · Gauss family

Bessel

Gauss HG — Kummer Confl. HG — Airy

Hermite-Weber

· Heun family ($\approx$ Painlevé family)

Double Confl. Heun

Heun — Confl. Heun — Tri-Confl. Heun — Reduced Tri-Confl. Heun

Bi-Confl. Heun

↓ w. apparent singular point + Isomondromic deformation

$P_{III}$

$P_{VI}$ — $P_V$ — $P_{II}$ — $P_I$

$P_{IV}$

- Garnier families, Kawakami-Nakamura-Sakai families, etc, etc ⋯

# WHAT IS CONFLUENCE (UNFOLDING)?

**Too naive way**

$$Y' = \Big(\underbrace{\frac{A_{k+1}}{z^{k+1}} + \cdots + \frac{A_1}{z}}_{} + \cdots\Big) Y$$

perturbation ⇝ ↑ irregular singularity of Poincaré rank $k$

$$\widetilde{Y}' = \frac{A_{k+1}}{(z-c_0)(z-c_1)\cdots(z-c_k)} + \frac{A_k}{(z-c_0)\cdots(z-c_{k-1})} + \cdots + \frac{A_1}{z-c_0} + \cdots$$

$$= \frac{\widetilde{\hat{A}_0}}{z-c_0} + \frac{\widetilde{\hat{A}_1}}{z-c_1} + \cdots + \frac{\widetilde{\hat{A}_k}}{z-c_k} + \cdots \qquad (c_i \neq c_j)$$

Generic $\vec{c} = (c_0, \cdots, c_k)$ —confluence→ special $\vec{c}$ ($c_i = c_j$ for some $i, j$)

Fuchsian ODE ←unfolding— Irreg. sing. ODE

**Why too naive?**

- This naive confluence may NOT preserve many invariants
  - index of rigidity (= dim of Moduli space)
- Hopeless to trace the change of local invariants.
  - Hukuhara-Turrittin-Levelt normal form, Monodromy, Stokes str. ....

HOWEVER

Gauss, Heun, Painlevé, Garnier, K-N-S, ..., families DO preserve many invariants & share many analytic properties inside each family.

# PROTOTYPE (want to generalize this!)

Oshima's rigid families (Publ. RIMS 2021)

$\cup$

{Gauss family, ${}_pF_q$-families, other known rigid families}

# OUR STRATEGY

- Construct

$$\pi : \mathcal{M} \longrightarrow D \subset \mathbb{C}^n : \text{holomorphic family}$$

of Moduli spaces of Merom. Conn. on $\mathbb{P}^1$ with unramified irreg. sing. via unfolding of these sing pts.

- Then a local section of $\pi$ gives a family of ODE.

Previous Work

Inaba (Bull. Sci. Math. 2019) ← Further, confluence of isomonodromic deform is discussed

(nonresonant uramified irrg. sing case)

↑ we will remove this

↑ We do not discuss in this talk.

# §1 Moduli space of meromorphic connections on trivial bundle on $\mathbb{P}^1$ with unramified irregular singularities (after Boalch)

Truncated orbit of HTL-normal form

Set

$$\mathbb{C}[z]_k := \mathbb{C}[z]/\langle z^{k+1}\rangle, \quad \mathbb{C}[z^{-1}]_k := z^{-(k+1)}\mathbb{C}[[z]]/\mathbb{C}[[z]] \subset \mathbb{C}((z))/\mathbb{C}[[z]],$$

then $\mathbb{C}[z^{-1}]_k$ is a $\mathbb{C}[z]_k$-module.

Take

$$H = \left(\frac{H_k}{z^k} + \cdots + \frac{H_1}{z} + R\right)\frac{dz}{z} : \text{Hukuhara-Turrittin-Levelt normal form}$$

($H_i \in M_n(\mathbb{C})$ : diagonal, $R \in M_n(\mathbb{C})$ : $[H_i, R] = 0 \ \forall i$ ).

then we regard

$$H \in M_n(\mathbb{C}[z^{-1}]_k)dz.$$

Set

$$G_k := GL_n(\mathbb{C}[z]_k), \quad \mathfrak{g}_k = \mathrm{Lie}\, G_k \simeq M_n(\mathbb{C}[z]_k)$$

then

$$\mathfrak{g}_k^* \simeq M_n(\mathbb{C}[z^{-1}]_k)dz \quad \text{via}$$

$$\begin{array}{ccc} M_n(\mathbb{C}[z]_k) \times M_n(\mathbb{C}[z^{-1}]_k)dz & \longrightarrow & \mathbb{C} \\ X \quad\quad Y dz & \longmapsto & \operatorname{Res}_{z=0}(\mathrm{tr} XY)dz \end{array}$$

**Def.** (truncated orbit)

$$\mathcal{O}_H := \mathrm{Ad}^*(G_k)(H)$$

**Def.** (moduli space of connection on triv. bdl. on $\mathbb{P}^1$)

$\mathbb{H} = (H_0, H_1, \cdots, H_r)$ : $(r+1)$-tuple of HTL-normal forms of size $n$.

$$\mathcal{M}(\mathbb{H}) := \left\{ d - \left( \sum_{i=1}^{r} \sum_{\nu=0}^{k_i} \frac{A_\nu^{(i)}}{(z-a_i)^{\nu+1}} - \sum_{\nu=1}^{k_0} A_\nu^{(0)} z^{\nu-1} \right) dz \;\middle|\; \text{irreducible},\ \left( \sum_{\nu=0}^{k_i} \frac{A_\nu^{(i)}}{z^\nu} \right) \frac{dz}{z} \in \mathcal{O}_{H_i} \right\} \Big/ GL_n(\mathbb{C}).$$

Here $A_0^{(0)} := -\sum_{i=1}^{r} A_0^{(i)}$ and

irreducible $\overset{\text{def.}}{\Longleftrightarrow}$ $(A_\nu^{(i)})_{\substack{i=0,\cdots,n \\ \nu=0,\cdots,k_i}}$ has no nontrivial simultaneous invariant subspace of $\mathbb{C}^n$.

<u>KNOWN</u>

$\mathcal{M}(\mathbb{H})$ is a smooth & connected complex symplectic manifold if $\neq \emptyset$.

# § 2 Deformation of $\mathcal{O}_H$

## Deformation of HTL-normal form

$H$ : HTL-normal form as above.

$$H(\vec{c}) = \left( \frac{H_k}{(z-c_1)\cdots(z-c_k)} + \frac{H_{k-1}}{(z-c_1)\cdots(z-c_{k-1})} + \cdots + \frac{H_1}{z-c_1} + R \right) \frac{dz}{z}$$

Then

$$H(0) = H$$

For $\forall \vec{c} \in \mathbb{C}^{k+1}$, $\exists\, c_{i_1}, \cdots, c_{i_\ell} \in \{c_0, c_1, \cdots, c_k\}$ s.t.

$$H(\vec{c}) = \sum_{j=1}^{\ell} \underbrace{\sum_{\nu=1}^{k_j} \frac{\widetilde{H}_\nu^{(j)}}{(z-c_{i_j})^{\nu+1}} dz}.$$

In this case, setting

$$H^{(j)} := \sum_{\nu=1}^{k_j} \frac{H_\nu^{(j)}}{z^\nu} \frac{dz}{z}, \quad \text{(HTL-normal form again!)}$$

we define

$$\mathcal{O}_{H(\vec{c})} := \left\{ \sum_{j=1}^{\ell} \sum_{\nu=1}^{k_j} \frac{A_\nu^{(j)}}{(z-c_{i_j})^{\nu+1}} dz \;\middle|\; \sum_{\nu=1}^{k_j} \frac{A_\nu^{(j)}}{z^\nu} \in \mathcal{O}_{H^{(j)}} \;\; \forall j \right\}$$

$$\simeq \prod_{j=1}^{\ell} \mathcal{O}_{H^{(j)}}$$

THM. $\mathcal{O}_H$ : complex manifold.

$0 \in \exists D \subset \mathbb{C}^n$ : open dense

$\exists\, \pi : \mathcal{O}_H \longrightarrow D$ : holomorphic surjective submersion.

s.t.

$$\pi^{-1}(\vec{c}) \simeq \mathcal{O}_{H(\vec{c})} \qquad \forall \vec{c} \in D$$

# Triangular decomposition of $\mathcal{O}_H$

Consider

$$\mathbb{C}^n = \bigoplus_{i=1}^{m_k} V_{[k,i]} > \bigoplus_{i=1}^{m_{k-1}} V_{[k-1,i]} > \cdots > \bigoplus_{i=1}^{m_1} V_{[1,i]},$$

(↑ refinement)

$\bigoplus_{i=1}^{m_j} V_{[j,i]}$ : simultaneous eigenspace decomposition of $\mathbb{C}^n$ by $(H_k, H_{k-1}, \cdots, H_j)$

Set

$\mathfrak{l}_j, \mathfrak{n}_j, \mathfrak{u}_j \subset M_n(\mathbb{C})$ : block mtx. along $\bigoplus_{i=1}^{m_j} V_{[j,i]}$

diag. / strict upper triangle / strict lower triange

$$L_j(\mathbb{C}[z]_j) := G_j \cap \mathfrak{l}_j(\mathbb{C}[z]_j)$$

$$P_j(\mathbb{C}[z]_j) := G_j \cap (\mathfrak{l}_j \oplus \mathfrak{n}_j)(\mathbb{C}[z]_j)$$

<u>THM</u> (H-Yamakawa, H.)

$$\mathcal{O}_H^{(j+1)} \simeq T^* G_j / P_j(\mathbb{C}[z]_j) \times \mathcal{O}_H^{(j)}$$

for $j = 1, 2, \cdots, k$.

Here

$$\mathcal{O}^{(j)}_H := Ad^*(L_j(\mathbb{C}[Z]_j))(H), \quad j = 1, 2, \cdots, k.$$

$$\mathcal{O}^{(k+1)}_H := \mathcal{O}_H.$$

<u>Deformation of $\mathbb{C}[Z]_j$, $\mathbb{C}[Z^{-1}]_j dz$</u>

For $\vec{c} = (c_1, \cdots, c_j) \in \mathbb{C}^j$, define a divisor on $\mathbb{C}$ by

$$D(\vec{c})_j := 0 + c_1 + \cdots + c_j.$$

$\Omega(D(\vec{c})_j) := \Omega_{D(\vec{c})_j}(\mathbb{P}^1)$ : set of merom 1-forms $\varphi$ s.t. $\mathrm{div}(\varphi) \geq -D(\vec{c})_j$.

$\hat{\Omega}(D(\vec{c})_j) := \Omega(D(\vec{c})_j) / \mathbb{C}[Z]dz$

$L(-D(\vec{c})_j) := \mathcal{O}_{-D(\vec{c})_j}(\mathbb{P}^1)$ : set of rational functions $f$ s.t. $\mathrm{div}(f) \geq D(\vec{c})_j$

$\check{L}(-D(\vec{c})) := \mathbb{C}[Z] / L(-D(\vec{c}))$

<u>Lem</u>

- $\hat{\Omega}(D(\vec{c})_j)$ is a $\check{L}(-D(\vec{c})_j)$-module.
- $\hat{\Omega}(D(0)_j) \simeq \mathbb{C}[Z^{-1}]_j dz$, $\check{L}(-D(0)_j) \simeq \mathbb{C}[Z]_j$.

<u>Def.</u> (residue map)

$$\mathrm{res} : \hat{\Omega}(D(\vec{c})_j) \ni f dz \longmapsto \oint_R f dz \in \mathbb{C}$$

R : a circle surrounding $0, c_1, \cdots, c_j$.

Under the following identifications

$$\check{L}(-D(c);j) \cong \{ g_0 + g_1(z-c_j) + g_2(z-c_j)(z-c_{j-1}) + \cdots + g_j(z-c_j)\cdots(z-c_1) \mid g_i \in \mathbb{C} \}$$

$$\hat{\Omega}(D(c);j) \cong \left\{ \left( f_0 + \frac{f_1}{z-c_1} + \frac{f_2}{(z-c_1)(z-c_2)} + \cdots + \frac{f_j}{(z-c_1)\cdots(z-c_j)} \right) \frac{dz}{z} \mid f_i \in \mathbb{C} \right\},$$

we define a pair of families

$$\check{L}(-D(S);j) := \{ g_0 + g_1(z-c_j) + g_2(z-c_j)(z-c_{j-1}) + \cdots + g_j(z-c_j)\cdots(z-c_1) \mid g_i \in \mathbb{C},\ \vec{c} \in S \} \simeq \mathbb{C}^{j+1} \times S$$

$$\hat{\Omega}(D(S);j) := \left\{ \left( f_0 + \frac{f_1}{z-c_1} + \frac{f_2}{(z-c_1)(z-c_2)} + \cdots + \frac{f_j}{(z-c_1)\cdots(z-c_j)} \right) \frac{dz}{z} \mid f_i \in \mathbb{C},\ \vec{c} \in S \right\} \simeq \mathbb{C}^{j+1} \times S$$

for $0 \in S \underset{\text{open}}{\subset} \mathbb{C}^j$.

- Then $\check{L}(-D(S);j)$, $\hat{\Omega}(D(S);j)$ are deformations of $\mathbb{C}[z]_j$, $\mathbb{C}[z^{-1}]_j$.
- Fiberwise action $\check{L}(-D(S);j) \curvearrowright \hat{\Omega}(D(S);j)$ gives a holomorphic transformations on $\hat{\Omega}(D(S);j)$.

Caution!

$\pi : GL_n(\check{L}(-D(S)_j)) \to S$ has no nontrivial global section!

$\det g(\vec{c})$ $(g(\vec{c}) \in M_n(\check{L}(-D(S))))$ may have zero as a function of $\vec{c} \in S$.

But

for $n(\vec{c}) \in N(\check{L}(-D(S)_j))$ (← unipotent mtx.), $\det n(\vec{c}) = 1$ (constant!).

- Bruhat decomposition of $G_j / P_j(\mathbb{C}[Z]_j)$ —

$N^+, N^- \subset GL_n(\mathbb{C})$ : subgroup of upper (lower) unpotent mtrices.

$W \simeq \mathfrak{S}_n$ : Weyl group of $GL_n(\mathbb{C})$

$N_\omega' := N^+ \cap \omega N^- \omega^{-1}$ $(\omega \in W)$

$W_j \subset W$ : subgroup associated to the parabolic subalgebra $\mathfrak{p}_j := \mathfrak{l}_j \oplus \mathfrak{n}_j$

THM (H)

$\{w_1, \cdots, w_t\}$ : a complete system of representatives of $W/W_j$

Then

$$GL_n(\mathbb{C}[Z]_j) = \coprod_{i=1}^{t} (\widehat{N_j})_{\omega_i}(\mathbb{C}[Z]_j)\ \omega_j\ P_j(\mathbb{C}[Z]_j)$$

where

$$(\widehat{N_j})_{\omega_i}(\mathbb{C}[Z])_j = \{ n(z) \in N_j(\mathbb{C}[Z]_j) \mid n(0) \in N_{\omega_i}' \}$$

## Deformation of $T^*G_j/P_j(\mathbb{C}[z]_j)$ —

Take the following trivialization

$$T^*G_j/P_j(\mathbb{C}[z]_j) \simeq G_j/P_j(\mathbb{C}[z]_j) \times \mathfrak{n}_j^*(\mathbb{C}[z^{-1}]_j\, dz)$$

Then define for $\mathcal{S} \subset \mathbb{C}^j$,

$$T^*G_j/P_j(\mathcal{S}) := \coprod_{i=1}^{t} (\hat{N}_j)_{\omega_i}(\check{L}(-D(\mathcal{S})_j)) \times \mathfrak{n}_j^*(\iota\hat{\Omega}(D(\mathcal{S})_j))$$

where

$$(\hat{N}_j)_{\omega_i}(\check{L}(-D(\mathcal{S})_j)) := \{ n(z) \in N_j(\check{L}(-D(\mathcal{S})_j)) \mid n(c_j) \in N_{\omega_i}' \}.$$

$\rightsquigarrow$

The decomposition $\mathcal{O}_H^{(j+1)} \simeq T^*G_j/P_j(\mathbb{C}[z]_j) \times \mathcal{O}_H^{(j)}$ & the deformation $T^*G_j/P_j(\vec{c})$ enable us to define a deformation of the truncated orbit $\mathcal{O}_H$:

$$\pi : \mathbb{O}_H \longrightarrow D \subset \mathbb{C}^{k+1}$$

s.t.

$$\pi^{-1}(\vec{c}) \simeq \mathcal{O}_{H(\vec{c})} \text{ for } \vec{c} \in D.$$

## §.3 Deformation of $\mathcal{M}(\mathbb{H})$

$\mathbb{H} = (H_0, H_1, \cdots, H_r)$ : $(r+1)$-tuple of HTL-normal forms of size $n$.

$H_i(\vec{c}_i)$ $(\vec{c}_i \in \mathbb{C}^{k_i+1})$ : deformations of $H_i$ as above.

$\pi_i : \mathbb{O}_{H_i} \to D_i \subset \mathbb{C}^{k_i+1}$ : deformations of $\mathbb{O}_{H_i}$ as above.

$$\mathrm{res} : \prod_{i=0}^{r} \mathbb{O}_{H_i} \to M_n(\mathbb{C})$$
$$(A_i) \mapsto \sum_{i=0}^{r} \mathrm{res}(A_i)$$

$$\mathcal{M} := \left\{ (A_i) \in \prod_{i=0}^{r} \mathbb{O}_{H_i} \;\middle|\; \begin{array}{l} \mathrm{res}(A_i) = 0 \\ \text{irreducible} \end{array} \right\} \Big/ GL_n(\mathbb{C})$$

$$\downarrow \pi$$

$$\prod_{i=0}^{r} D_i$$

<u>THM.</u> If $\mathcal{M}(\mathbb{H}) \neq \phi$, then

1) $\mathcal{M}$ is a smooth complex manifold

2) $\pi : \mathcal{M} \to \prod_{i=0}^{r} D_i$ is holomorphic, surjective, submersive. (i.e. flat as analytic spaces)

3) $\pi^{-1}(\vec{c}) \simeq \mathcal{M}(\mathbb{H}(\vec{c}))$ $\forall \vec{c} \in \prod_{i=0}^{r} D_i$

These family contain Heun (Painlevé), Garnier, K-N-S, ..., many known families arising from confluence!

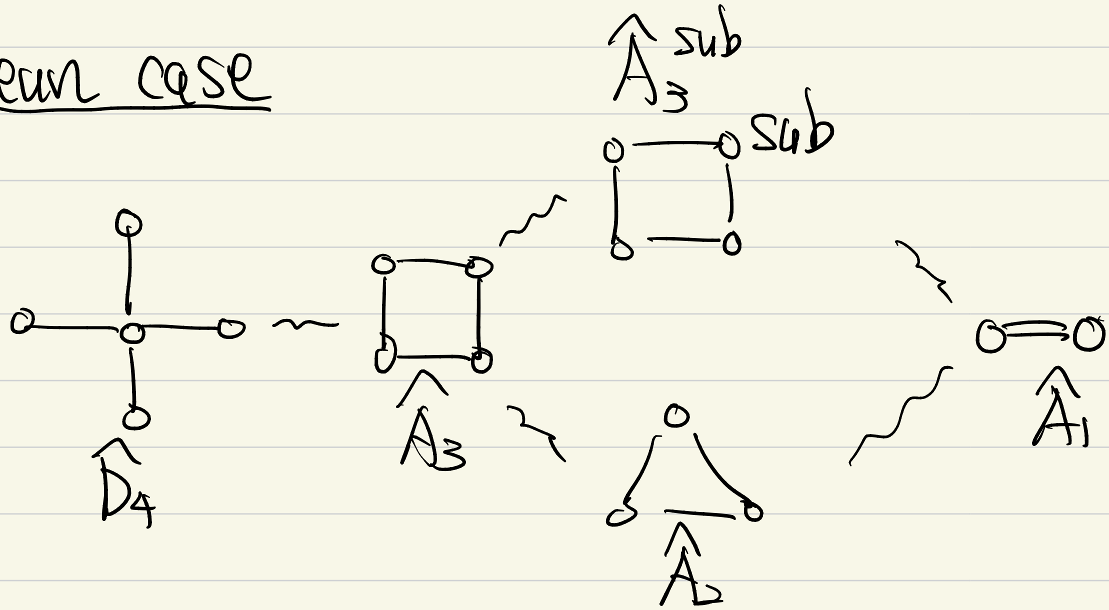
the hardest part to show is 2).

<u>Key</u> Solve simultaneous additive Deligne-Simpson problem by using quiver theory!

$\mathcal{M}(\mathcal{H})$ moduli of connection $\longleftrightarrow$ $\mathfrak{M}_\lambda(Q, \alpha)$ quiver variety

(Crawley-Boevey, Boalch, Yamakawa, H-Yamakawa, H.)

<u>Heun case</u>



THM (H)

$Q$ : quiver associated to $\mathcal{M}(\mathcal{H})$

$Q^{gen}$ : quiver associated to $\mathcal{M}(\mathcal{H}(\vec{c})) \simeq \pi^{-1}(\vec{c})$ for generic fiber

$\exists$ $\mathrm{rep}(Q) \hookrightarrow \mathrm{rep}(Q^{gen})$ : fully faithful functor

$\rightsquigarrow$

$\mathcal{M}(\mathcal{H}) \neq \emptyset \iff \mathcal{M}(\mathcal{H}(\vec{c})) \neq \emptyset$

$\rightsquigarrow$ $\pi$ is surjective & submersive.