Confluent approach to Fifth Painlevé equation

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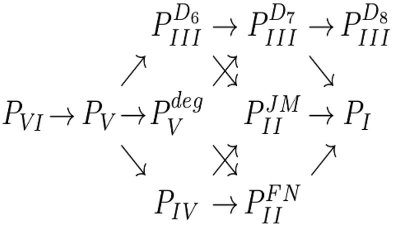
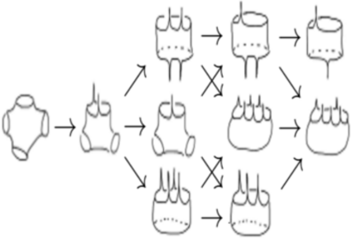
Klimeš, M., *Wild Monodromy of the Fifth Painlevé Equation and its Action on Wild Character Variety: An Approach of Confluence*, arXiv:1609.05185

Confluence



(a) Painlevé equations



(b) Isomonodromic deformations

[Ohyama, Okumura (2006), Chekhov, Mazzocco, Roubtsov (2017)]

Goal: transfer knowledge along the diagram

Problem: divergence on both sides of the Riemann–Hilbert correspondence

Nonlinear Monodromy

Okamoto–Painlevé system:

$$P_J: \quad \frac{dq}{dt} = \frac{\partial H_J(q,p,t)}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H_J(q,p,t)}{\partial q}, \qquad J = I, \dots, VI,$$

leaves are transverse to the fibration $(q,p,t) \mapsto t$.

Okamoto's completion of the phase space:

1. leaves transverse to each fiber $\mathcal{M}_{J,t}$ = *Okamoto's space of initial conditions*, endowed with the symplectic form $\omega = dq \wedge dp$,
2. *Geometric Painlevé property:* every leaf is a covering of $\mathbb{CP}^1 \setminus \mathrm{Sing}(P_J)$.

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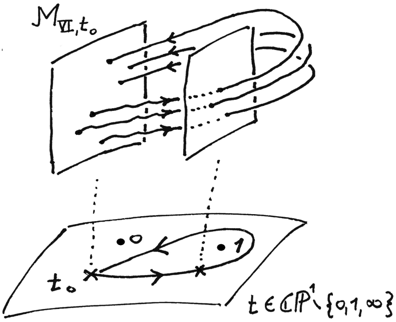
Monodromy (nonlinear):

$$\pi_1(\mathbb{CP}^1 \setminus \mathrm{Sing}(P_J), t_0) \to \mathrm{Aut}_\omega(\mathcal{M}_{J,t_0})$$



P_{VI}: $\mathrm{Sing}(P_{VI}) = \{0, 1, \infty\}$

- represented by explicit algebraic action of a braid group on a character variety [Dubrovin, Mazzocco (2000), Iwasaki (2002)]
- algebraic solutions correspond to finite orbits [Hitchin (1995), Dubrovin, Mazzocco (2000), Boalch (2003-2010), Lysovyy, Tykhyy (2014)]
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Conjecture (Ramis (2012))

1. *Painlevé property extends to "wild Painlevé property" (Ecalle's resurgence?) and monodromy extends to a "wild monodromy" (nonlinear Stokes phenomenon at irregular singularities),*
2. *Galoisian significance of the wild monodromy (Malgrange–Galois pseudogroup),*
3. *Riemann–Hilbert correspondence algebrizes everything (wild monodromy dynamics on wild character variety is rational and depends rationally on the parameters).*

Non-linear Stokes phenomenon of P_V

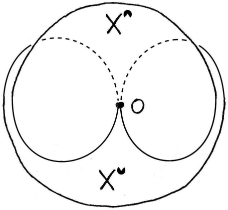
The Okamoto–Painlevé system of (non-degenerate) P_V near $(q, p, t) = (0, 0, \infty)$:

$$(*) \qquad P_V: \quad x^2 \frac{dq}{dx} = -\frac{\partial H_V(q,p,x^{-1})}{\partial p}, \qquad x^2 \frac{dp}{dx} = \frac{\partial H_V(q,p,x^{-1})}{\partial q}, \qquad x = t^{-1}$$

Theorem (Takano (1983), Shimomura (1983))

There exists a pair of sectorial transversely symplectic transformations $\binom{q}{p} = \Psi^\bullet(u, x)$, $|u| < \delta$, $x \in X^\bullet$, $\bullet = ◠, ◡$, *bringing* $(*)$ *to a formal normal form*

$$x^2 \frac{d}{dx} u = \Big(1 - (2\vartheta_0 + \tilde{\vartheta}_1 - 1)x + 4x\, u_1 u_2\Big)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u.$$



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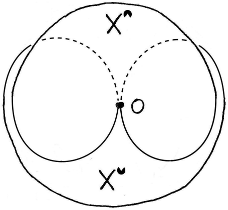

Canonical 2-parameter family of solutions

$$\binom{q^\bullet}{p^\bullet}(x; c) = \Psi^\bullet(\cdot, x) \circ u^\bullet(x; c), \qquad u^\bullet(x; c) = \begin{pmatrix} c_1\, e^{-\frac{1}{x}} x^{-(2\vartheta_0 + \tilde{\vartheta}_1 - 1) + 4c_1 c_2}, \\ c_2\, e^{\frac{1}{x}} x^{(2\vartheta_0 + \tilde{\vartheta}_1 - 1) - 4c_1 c_2} \end{pmatrix},$$

$c \in (\mathbb{C}^2, 0)$ ... local coordinate on the space of leaves over $X^\bullet$, $\bullet = \frown, \smile$.

$c = 0$: *sectorial center manifold solution* (pole-free on $X^\bullet$).

Nonlinear wild monodromy pseudogroup of P_V

Acts locally on leaves. Generated by:

- *Exponential torus*: commutative Lie group of analytic symplectic symmetries of the formal normal form

$$\boldsymbol{T}_{\alpha}(c) = \begin{pmatrix} e^{\alpha(c_1 c_2)} c_1 \\ e^{-\alpha(c_1 c_2)} c_2 \end{pmatrix}, \qquad \alpha \in \mathcal{O}(\mathbb{C}, 0),$$

Formal monodromy $u^{\bullet}(e^{2\pi i} x; c) = u^{\bullet}(x; \boldsymbol{N}(c))$,

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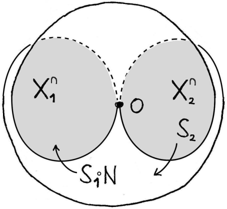
- *Stokes operators*:

$$\begin{pmatrix} q^{\cup} \\ p^{\cup} \end{pmatrix}(x; c) = \begin{pmatrix} q^{\cap} \\ p^{\cap} \end{pmatrix}(x; \boldsymbol{S}_1 \circ \boldsymbol{N}(c)), \qquad x \in X_1^\cap,$$

$$\begin{pmatrix} q^{\cap} \\ p^{\cap} \end{pmatrix}(x; c) = \begin{pmatrix} q^{\cup} \\ p^{\cup} \end{pmatrix}(x; \boldsymbol{S}_2(c)), \qquad x \in X_2^\cap.$$



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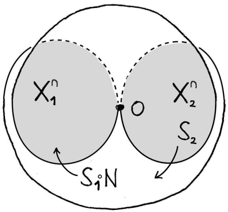

- *Monodromy*:

$$\begin{pmatrix} q^{\frown} \\ p^{\frown} \end{pmatrix}(e^{2\pi i}x;c) = \begin{pmatrix} q^{\frown} \\ p^{\frown} \end{pmatrix}(x; \boldsymbol{M}^{\frown}(c)), \qquad \boldsymbol{M}^{\frown}(c) = \boldsymbol{S}_2 \circ \boldsymbol{S}_1 \circ \boldsymbol{N}(c),$$

The confluence $P_{VI} \to P_V$

$$t_{VI} = 1 + \epsilon t_V, \qquad \vartheta_{t,VI} = \tfrac{1}{\epsilon}, \quad \vartheta_{1,VI} = -\tfrac{1}{\epsilon} + \tilde{\vartheta}_{1,V}, \quad x = \tfrac{1}{t_V} + \epsilon.$$

Confluent system:

$$(*) \qquad x(x-\epsilon)\frac{dq}{dx} = -\frac{\partial H(q,p,x,\epsilon)}{\partial p}, \qquad x(x-\epsilon)\frac{dp}{dx} = \frac{\partial H(q,p,x,\epsilon)}{\partial q}.$$

Theorem (K. (2016))

There exist "sectorial" transversely symplectic transformations $\binom{q}{p} = \Psi^{\bullet}_{\pm}(u,x,\epsilon)$ *on a parametric family of domains* $|u| < \delta$, $x \in X^{\bullet}_{\pm}(\epsilon)$, $\epsilon \in E_{\pm}$, $\bullet = \frown, \smile$, *bringing* $(*)$ *to a confluent formal normal form*

$$x(x-\epsilon)\frac{du}{dx} = \Big(1 - \epsilon - (x-\epsilon)\vartheta_0 - x(\vartheta_0 + \tilde{\vartheta}_1 - 1) + 2(2x-\epsilon)u_1 u_2\Big)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u.$$

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where

$$u_\pm^\bullet(x,\epsilon;c) = \begin{pmatrix} c_1 E \\ c_2 E^{-1}\end{pmatrix}, \quad E(x,\epsilon;c_1c_2) = \begin{cases} x^{-\frac{1}{\epsilon}+1-\vartheta_0+2c_1c_2}(x-\epsilon)^{\frac{1}{\epsilon}-1-\vartheta_0-\tilde\vartheta_1+2c_1c_2}, \\ e^{-\frac{1}{x}}x^{-2\vartheta_0-\tilde\vartheta_1-1+4c_2c_2}, \qquad \epsilon = 0.\end{cases}$$

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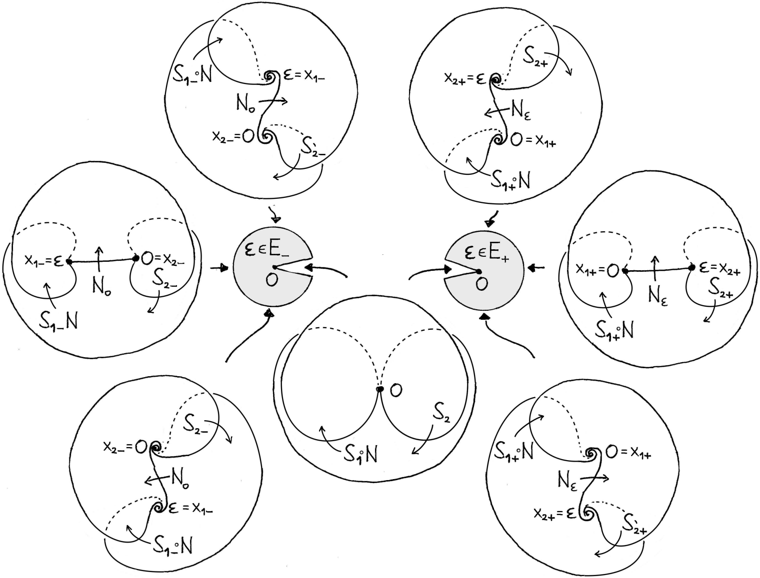

Unfolded Stokes operators $\boldsymbol{S}_{1,\pm}$, $\boldsymbol{S}_{2,\pm}$, *formal monodromy* $\boldsymbol{N}_0$, $\boldsymbol{N}_\epsilon$, $\boldsymbol{N}$.

Decomposition of monodromy

For $\epsilon \in \mathrm{E}_+ \smallsetminus \{0\}$, $x_0 \in \mathrm{X}_+^{\spadesuit}$, the *monodromy* around $x = 0$ and $x = \epsilon$ *decomposes* as:

$$\boldsymbol{M}_{0,+}^{\spadesuit} = \boldsymbol{N}_\epsilon^{\circ(-1)} \circ \boldsymbol{S}_{1,+} \circ \boldsymbol{N}, \qquad \boldsymbol{M}_{\epsilon,+}^{\spadesuit} = \boldsymbol{S}_{2,+} \circ \boldsymbol{N}_\epsilon,$$

where the *unfolded Stokes operators* $\boldsymbol{S}_{i,\pm}(c,\epsilon)$ tend to the non-linear Stokes operators $\boldsymbol{S}_i(c,0)$ of P_V when $\epsilon \to 0$, and the *formal monodromies*

$$\boldsymbol{N}_0(c,\epsilon) = \begin{pmatrix} e^{-\frac{2\pi i}{\epsilon} - 2\pi i(\vartheta_0 - 2c_1c_2)} c_1 \\ e^{\frac{2\pi i}{\epsilon} + 2\pi i(\vartheta_0 - 2c_1c_2)} c_2 \end{pmatrix}, \qquad \boldsymbol{N}_\epsilon(c,\epsilon) = \begin{pmatrix} e^{\frac{2\pi i}{\epsilon} - 2\pi i(\vartheta_0 + \tilde{\vartheta}_1 - 2c_1c_2)} c_1 \\ e^{-\frac{2\pi i}{\epsilon} + 2\pi i(\vartheta_0 + \tilde{\vartheta}_1 - 2c_1c_2)} c_2 \end{pmatrix},$$

belong to the exponential torus.

Accumulation of monodromy

Discretization along sequences $\{\epsilon_n\}_{n\in\pm\mathbb{N}} \subset \mathsf{E}_\pm \setminus \{0\}$

$$\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n, \quad \text{s.t. } \kappa := e^{\frac{2\pi i}{\epsilon}} \in \mathbb{C}^* \text{ is constant.}$$

Accumulation to a 1-parameter family depending on κ of *wild monodromy operators*

$$\boldsymbol{M}^{\frown}_{0,+}(c,\epsilon_n) \to \tilde{\boldsymbol{M}}^{\frown}_{0,+}(c;\kappa) = \tilde{\boldsymbol{N}}_\epsilon(\cdot;\kappa)^{\circ(-1)} \circ \boldsymbol{S}_1 \circ \boldsymbol{N}(c),$$

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where

$$\tilde{\boldsymbol{N}}_0(c;\kappa) = \begin{pmatrix} \frac{1}{\kappa} e^{2\pi i(-\vartheta_0 + 2c_1c_2)} c_1 \\ \kappa e^{2\pi i(\vartheta_0 - 2c_1c_2)} c_2 \end{pmatrix}, \qquad \tilde{\boldsymbol{N}}_\epsilon(c;\kappa) = \begin{pmatrix} \kappa e^{2\pi i(-\vartheta_0 - \tilde{\vartheta}_1 + 2c_1c_2)} c_1 \\ \frac{1}{\kappa} e^{2\pi i(\vartheta_0 + \tilde{\vartheta}_1 - 2c_1c_2)} c_2 \end{pmatrix},$$

and $\boldsymbol{N} = \tilde{\boldsymbol{N}}_0(\kappa) \circ \tilde{\boldsymbol{N}}_\epsilon(\kappa)$, are elements of the exponential torus.

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The *Stokes operators* can be now expressed e.g. as

$$\boldsymbol{N}^{\circ(-1)} \circ \boldsymbol{S}_1 \circ \boldsymbol{N}(c) = \tilde{\boldsymbol{M}}^{\frown}_{0,+}(c;\kappa)\Big|_{\kappa = e^{2\pi i(-\vartheta_0 + 2c_1c_2)}}, \quad \boldsymbol{S}_2(c) = \tilde{\boldsymbol{M}}^{\frown}_{\epsilon,+}(c;\kappa)\Big|_{e^{2\pi i(\vartheta_0 + \tilde{\vartheta}_1 - 2c_1c_2)}},$$

while the *infinitesimal "generator"* $(c_1\partial_{c_1} - c_2\partial_{c_2})$ *of the exponential torus* is given by

$$\dot{c} = -\left(\kappa \tfrac{d}{d\kappa} \tilde{\boldsymbol{M}}^{\frown}_{0,+}(\cdot;\kappa)\right) \circ \left(\tilde{\boldsymbol{M}}^{\frown}_{0,+}(c;\kappa)\right)^{\circ(-1)}.$$

Character variety of P_{VI}

Flat traceless meromorphic connection on the trivial bundle on $\mathbb{CP}^1$

$$\nabla_{VI} = d - \left[\frac{A_{\mathbf{0}}(t)}{z} + \frac{A_t(t)}{z-t} + \frac{A_{\mathbf{1}}(t)}{z-\mathbf{1}}\right] dz + \frac{A_t(t)}{z-t} dt,$$

fixed parameters: $\pm\frac{\vartheta_l}{2}\ldots$ eigenvalues of the residue matrices A_l, $l = 0, t, 1, \infty$.

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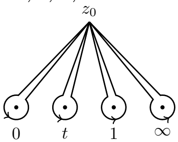
(Linear) monodromy representation:

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}, z_0) \to SL_2(\mathbb{C}),$$



$\rho(\gamma_l) = M_l$, $l = 0, t, 1, \infty$, have eigenvalues $e_l = e^{\pi i \vartheta_l}$, e_l^{-1}.

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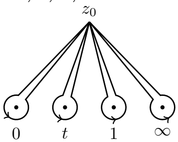

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Trace coordinates on the monodromy manifold:

$$a_l = e_l + e_l^{-1} = \mathrm{tr}(M_l),\ l = 0, t, 1, \infty, \quad X_i = \mathrm{tr}(M_j M_k),\ \{i, j, k\} = \{0, t, 1\}.$$

Fricke relation

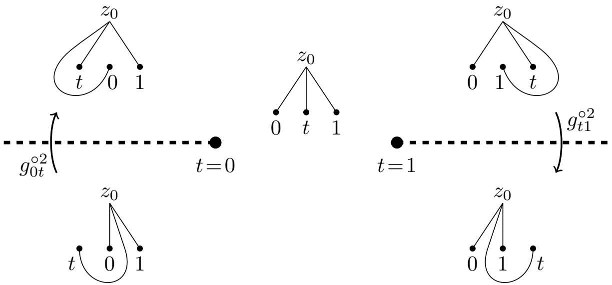
$$F(X, a) := X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - \theta_0 X_0 - \theta_t X_t - \theta_1 X_1 + \theta_\infty = 0,$$

with $\theta_i = a_i a_\infty + a_j a_k$, $i = 0, t, 1$, and $\theta_\infty = a_0 a_t a_1 a_\infty + a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - 4$.

The character variety of P_{VI}: $\mathcal{S}_{VI}(a) = \{X \in \mathbb{C}^3 : F(X, a) = 0\}$.

Symplectic form $\omega_{\mathcal{S}_{VI}} = \frac{dX_j \wedge dX_i}{2\pi i F_k}$, (i, j, k) cyclic permutation of $(0, t, 1)$.

Non-linear monodromy action on the character variety of P_{VI}



Theorem (Dubrovin, Mazzocco (2000), Iwasaki (2002))

Action of pure braid group on $\pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}, z_0)$ induces symplectic action on $\mathcal{S}_{VI}(\theta)$ which fixes the singularities of $\mathcal{S}_{VI}(\theta)$, and whose restriction on the smooth locus represents faithfully the nonlinear monodromy of P_{VI}.

$$g_{ij}^{\circ 2} : X_i \mapsto X_i - F_i + X_k F_j, \qquad F_j := \frac{dF}{dX_j},$$

$$X_j \mapsto X_j - F_j,$$

$$X_k \mapsto X_k,$$

Lines on $\mathcal{S}_{VI}$

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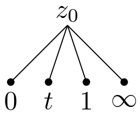
Assuming each M_l diagonalizable, $l = 0, t, 1, \infty$, it determines 2 invariant subspaces of the solution space, associated to the eigenvalues e_l, e_l^{-1}.

Theorem (K.)

Each pair $\{M_l, M_m\}$ gives rise to 4 different mixed bases (up to rescaling).
Associated are 4 lines, each corresponds to degeneracy of one mixed basis.



Lines on $\mathcal{S}_{VI}$

$\mathcal{S}_{VI}$ is a cubic surface with *24 lines* (counting multiplicity) + *3 lines at infinity*.
Explicit formulas [K.].

Assuming each M_l diagonalizable, $l = 0, t, 1, \infty$, it determines 2 invariant subspaces of the solution space, associated to the eigenvalues e_l, e_l^{-1}.

Theorem (K.)

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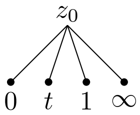

- Generalizes to $P_I, \ldots, P_V$.
- Intersection points: *special solutions* (pole-free on some "large" domains).

Joint project with E. Paul, J.-P. Ramis.

The confluence $P_{VI} \to P_V$

Change of variables $t \mapsto 1 + \epsilon t, \quad \vartheta_t \mapsto \frac{1}{\epsilon}, \quad \vartheta_1 \mapsto -\frac{1}{\epsilon} + \tilde{\vartheta}_1,$

Confluent family of connections depending on ϵ

$$\nabla_{conf} = d - \left[\frac{A_0(t)}{z} + \frac{A_1^{(0)}(t)}{(z-1)(z-1-\epsilon t)} + \frac{A_1^{(1)}(t)}{z-1}\right] dz + \frac{A_1^{(0)}(t)}{t(z-1-\epsilon t)} dt$$

$A_1^{(0)}$ has eigenvalues $\pm\frac{t}{2}$.

Theorem (Hurtubise, Lambert, Rousseau (2012 & 13), Parise (2001))

For $\epsilon \in \mathsf{E}_+$, resp. E_- (the same sectors as before!) a branch of the normalized mixed solution basis associated to the eigenvalues $e_1^{-1} = e^{\frac{\pi i}{\epsilon} - \pi i \tilde{\vartheta}_1}$ and $e_t = e^{\frac{\pi i}{\epsilon}}$, resp. e_1 and e_t^{-1}, converges as $\epsilon \to 0$ (uniformly on some domains $z \in \mathsf{Z}_\pm^\bullet(\epsilon)$) to each of the sectorial solution bases at the irregular singularity $z = 1$ for $\epsilon = 0$.

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Decomposition of (linear) monodromy: we restrict to $\epsilon \in \mathsf{E}_+$

$$M_t(\epsilon) = N_t(\epsilon)S_2(\epsilon) = \begin{pmatrix} e_t & e_t s_2 \\ 0 & \frac{1}{e_t} \end{pmatrix}, \qquad M_1(\epsilon) = S_1(\epsilon)N_1(\epsilon) = \begin{pmatrix} e_1 & 0 \\ e_1 s_1 & \frac{1}{e_1} \end{pmatrix},$$

where the *unfolded Stokes matrices* $S_1(\epsilon) = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}$, $S_2(\epsilon) = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}$ converge as $\epsilon \to 0$, while the *formal monodromies* $N_j = \begin{pmatrix} e_j & 0 \\ 0 & \frac{1}{e_j} \end{pmatrix}$ diverge.

Wild character variety

Coordinates on the *wild monodromy manifold* of P_V [van der Put, Saito (2009)]

$$a_0 = \mathrm{tr}(M_0) = e_0 + e_0^{-1}, \quad \tilde{e}_1 = e^{\pi i \tilde{\vartheta}_1} = e_t e_1, \quad a_\infty = \mathrm{tr}(M_\infty) = e_\infty + e_\infty^{-1},$$
$$\tilde{X}_0 = (M_0)_{22}, \quad \tilde{X}_1 = \mathrm{tr}(M_0 M_\infty), \quad \tilde{X}_\infty = (M_\infty)_{22}.$$

Same choice in the confluent situation.

Wild Fricke relation:

$$\tilde{F}(\tilde{X}, \tilde{\theta}) := \tilde{X}_0 \tilde{X}_1 \tilde{X}_\infty + \tilde{X}_0^2 + \tilde{X}_\infty^2 - \tilde{\theta}_0 \tilde{X}_0 - \tilde{\theta}_1 \tilde{X}_1 - \tilde{\theta}_\infty \tilde{X}_\infty + \tilde{\theta}_t = 0,$$

where $\tilde{\theta}_0 = a_0 + \tilde{e}_1 a_\infty, \quad \tilde{\theta}_1 = \tilde{e}_1, \quad \tilde{\theta}_\infty = a_\infty + \tilde{e}_1 a_0, \quad \tilde{\theta}_t = 1 + \tilde{e}_1 a_0 a_\infty + \tilde{e}_1^2.$

Wild character variety [Boalch (2014)]:

$$\mathcal{S}_V(\tilde{\theta}) = \{\tilde{X} \in \mathbb{C}^3 : \tilde{F}(\tilde{X}, \tilde{\theta}) = 0\}$$

Symplectic form: $\tilde{\omega}_{\mathcal{S}_V} = \frac{d\tilde{X}_i \wedge d\tilde{X}_j}{2\pi i \tilde{F}_k}$, (i, j, k) cyclic permutation of $(0, 1, \infty)$.

Confluence of character varieties

Theorem (K.)

For each $\epsilon \in \mathsf{E}_+ \setminus \{0\}$*, the change of coordinates on the monodromy manifold*

$$\left(\mathcal{S}_{VI}(\theta), \omega_{\mathcal{S}_{VI}}\right) \to \left(\mathcal{S}_{V}(\tilde{\theta}), \omega_{\mathcal{S}_{V}}\right)$$
$$X(\epsilon) \mapsto \tilde{X},$$

where $X(\epsilon)$ *are the trace coordinates, is a birational symplectic map (depending on* ϵ*), which is a blow-down of the line in* $\mathcal{S}_{VI}$ *that corresponds to degeneracy of the mixed basis of Hurtubise–Lambert–Rousseau.*

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Theorem/Conjecture (K., Paul, Ramis)

All the confluent degenerations of character varieties of Painlevé equations happen through a blow-down of a line.

Action of the nonlinear wild monodromy on $\mathcal{S}_V$

- transfer the braid group action from $\mathcal{S}_{VI}$ to $\mathcal{S}_V$,
- discretize: replace the divergent $e^{\frac{2\pi i}{\epsilon}}$ by a new parameter $\kappa \in \mathbb{C}^*$,
- express the action of the nonlinear Stokes operators and of the nonlinear exponential torus on the wild character variety of P_V.

Theorem (K.)

The nonlinear wild monodromy pseudogroup of P_V acts on $\mathcal{S}_V(\tilde{\theta})$ by birational symplectic transformations, which fix the singularities of $\mathcal{S}_V(\tilde{\theta})$, and whose restriction to the smooth locus of $\mathcal{S}_V(\tilde{\theta})$ represent it faithfully.

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i) The infinitesimal generator of the exponential torus corresponds to the Hamiltonian vector field $\frac{1}{\tilde X_0}\left(\tilde F_1 \frac{\partial}{\partial \tilde X_\infty} - \tilde F_\infty \frac{\partial}{\partial \tilde X_1}\right)$, *with* $H_0 = \frac{1}{2\pi i}\log \tilde X_0 + \frac{\vartheta_0}{2}$.
Its time-α-flow map is

$$
\begin{aligned}
\boldsymbol{t}(\,\cdot\,; e^{\alpha}): \quad \tilde X_0 &\mapsto \tilde X_0,\\
\tilde X_1 &\mapsto \tilde X_1 - (1-e^{-\alpha})\frac{\tilde F_\infty}{\tilde X_0} + (2-e^{\alpha}-e^{-\alpha})\frac{\tilde F_1}{\tilde X_0^2},\\
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ii) The formal monodromy acts as $\boldsymbol{n}(\tilde X) = \boldsymbol{t}(\tilde X; \frac{\tilde X_0^4}{\tilde e_1^2})$.

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iii) The Stokes operators act as $\boldsymbol{n}^{\circ(-1)} \circ \boldsymbol{s}_1 \circ \boldsymbol{n} : \ldots$, *and* $\boldsymbol{s}_2 : \ldots$.
iv) The total monodromy operator acts as

$$\begin{aligned} \boldsymbol{s}_2 \circ \boldsymbol{s}_1 \circ \boldsymbol{n} = g_{0\infty}^{\circ 2}:\quad \tilde{X}_0 &\mapsto \tilde{X}_0 - \tilde{F}_0, \\ \tilde{X}_1 &\mapsto \tilde{X}_1, \\ \tilde{X}_\infty &\mapsto \tilde{X}_\infty - \tilde{F}_\infty + \tilde{X}_1\tilde{F}_0. \end{aligned}$$