Quantum representation of Weyl group $W(E_8^{(1)})$

Web-seminar on Painlevé equations and related topics

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Based on the work [S.Moriyama-Y.Y. (arXiv:2104.06666v2[math:QA])]

Introduction

- **Algebraic differential eq $\leftrightarrow$ a field K with a derivation $'$.**

  e.g. The P_{IV} eq: $K = \mathbb{C}(p, q, t, a_1, a_2, \epsilon)$.

$$q' = 2pq - q^2 - qt - a_1, \quad p' = 2pq - p^2 + pt + a_2,$$
$$t' = \epsilon, \quad a_i' = 0, \quad \epsilon' = 0.$$

Non-autonomous Hamiltonian system with $H = pq(p - q - t) - a_1 p - a_2 q$.
If $\epsilon = 0$ (autonomous) $\to H(p, q)$ is conserved.

- **Bäcklund transformations $\in \mathrm{Aut}(K)$ commuting with $'$.**

  e.g. $\langle s_0, s_1, s_2, \pi \rangle = \tilde{W}(A_2^{(1)})$:

$$s_1 : \{p \to \frac{a_1}{q}, \quad a_1 \to -a_1, \quad a_2 \to a_1 + a_2\},$$
$$\pi : \{q \to -p, \quad p \to q - p + t, \quad a_1 \to a_2, \quad a_2 \to \epsilon - a_1 - a_2\},$$
$$s_2 = \pi s_1 \pi^{-1}, \quad s_0 = \pi s_2 \pi^{-1}.$$

(trivial actions are omitted)

- **Discrete eq** $\leftrightarrow T \in \mathrm{Aut}(K)$**.** iteration $\to$ dynamical system.

  e.g. d-P_{II} eq: $K = \mathbb{C}(p, q, t, a_1, a_2, \epsilon)$,

$$\boxed{\begin{aligned} T: \{q \to Q = p - q - t - \frac{a_2}{p}, \quad p \to -q - \frac{a_2}{p} + \frac{a_1 - \epsilon}{Q}, \\ a_1 \to a_1 - \epsilon, \quad a_2 \to a_2 + \epsilon\}. \end{aligned}}$$

  If $\epsilon \neq 0 \to$ non-autonomous system.

  If $\epsilon = 0 \to$ autonomous, $H(p, q)$ is conserved.

- **Symmetry =** $\mathrm{Aut}(K)$ **commuting with** T**.**

  e.g. Symmetry= $\langle r_0, r_1 \rangle = W(A_1^{(1)})$.

$$T = \pi^2 s_0 s_1 \text{ and } r_0 = s_0, r_1 = s_1 s_2 s_1 \in W(A_2^{(1)}).$$

The flow T and its symmetry $W(A_1^{(1)})$ are unified in a larger symmetry $W(A_2^{(1)})$.

$\to$ The full symmetry is of fundamental importance.

▲ **Affine Weyl group approach to Painlevé type eq** (e.g. [Noumi-Y (1998)])

- Pick a birational representation of affine Weyl group

$\to$ discrete flows $+$ its Bäcklund tr.

- This approach is useful in **quantum** setting as well.

▲ **Quantization**

Known representations are **birational symplectic**.

$\to$ **Natural to consider their quantization** through

$$\{p, q\} = 1 \quad \to \quad [p, q] = h \quad \text{or} \quad e^p e^q = e^h e^q e^p.$$

After AGT, the **quantum Painlevé equations** appear in various areas in math-phys.

• The main problem of the affine Weyl group approach is its initial step.

How can we find suitable birational reps?

• **Two methods** are known.

	Lie theory	rational surf
classical	Noumi-Y(2000)	Sakai(2001)
quantum	Kuroki(2011)	our problem

• **Lie theory.** Poisson actions of $W(\mathfrak{g})$ on $S(\mathfrak{n}_-)$ are formulated for Kac-Moody alg $\mathfrak{g}$. Their good quantization exists for $U(\mathfrak{g})$ and also for $U_q(\mathfrak{g})$. These are applicable for $E_8^{(1)}$, but huge in general.

• **Rational surface.** The Cremona isometry of rational surfaces give birational reps including $E_8^{(1)}$. Its quantization is our target.

For that, a result on $D_5^{(1)}$ [Hasegawa (2007)] gives an important hint.

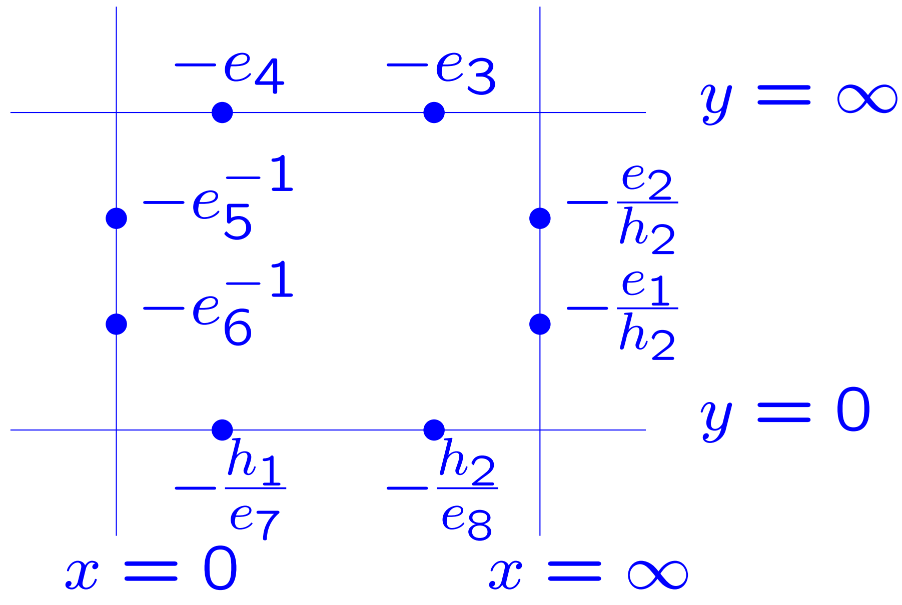
▲ Plan of the talk

1. $D_5^{(1)}$ example
2. The representation of $W(E_8^{(1)})$
3. Lifting the representation including τ variables
4. F polynomials and the quantum curve

1. $D_5^{(1)}$ example

▲ The geometry:

- Let $X = X_{(h_i, e_i)}$ be a blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 8 points sitting on 4 lines.



Picard group $\mathrm{Pic}(X)$ is generated by H_1, H_2, E_1, ..., E_8.

- The **affine Weyl group** $W(D_5^{(1)})$.

$$\left\langle \begin{array}{ccccccc} & & s_0 & & s_4 & & \\ & & | & & | & & \\ s_1 & - & s_2 & - & s_3 & - & s_5 \end{array} \;\middle|\; \begin{array}{ll} s_i^2 = 1, & \\ s_i s_j = s_j s_i, & (s_i \quad s_j), \\ s_i s_j s_i = s_j s_i s_j, & (s_i - s_j). \end{array} \right\rangle$$

$W(D_5^{(1)})$ acts on X (birationally on $\mathbb{P}^1 \times \mathbb{P}^1$).

- The explicit actions s_i on $K = \mathbb{C}(h_1, h_2, e_1, \dots, e_8, x, y)$:

$$
\boxed{
\begin{aligned}
&s_0 = \{e_7 \leftrightarrow e_8\}, \quad s_1 = \{e_3 \leftrightarrow e_4\},\\
&s_2 = \{e_3 \to \frac{h_1}{e_7},\ e_7 \to \frac{h_1}{e_3},\ h_2 \to \frac{h_1 h_2}{e_3 e_7},\ y \to \frac{1+\frac{e_7}{h_1}x}{1+\frac{x}{e_3}}y\},\\
&s_3 = \{e_1 \to \frac{h_2}{e_5},\ e_5 \to \frac{h_2}{e_1},\ h_1 \to \frac{h_1 h_2}{e_1 e_5},\ x \to x\frac{1+\frac{h_2}{e_1}y}{1+e_5 y}\},\\
&s_4 = \{e_1 \leftrightarrow e_2\}, \quad s_5 = \{e_5 \leftrightarrow e_6\}.
\end{aligned}
}
$$

- Actions on $\{h_i, e_i\}$ are the standard **'linear'** reflections on Pic(X)

(written in multiplicative variables: $h_i = e^{H_i}, e_i = e^{E_i}$).

$\rightarrow$ The actions on x,y are its **natural birational lift** to $\mathbb{P}^1 \times \mathbb{P}^1$.

- The Weyl group relations hold true also when x, y are **non-comutative**

[Hasegawa(2007)].

• Check of the Weyl group relations.

From $s_2 = \{e_3 \to \frac{h_1}{e_7},\ e_7 \to \frac{h_1}{e_3},\ h_2 \to \frac{h_1 h_2}{e_3 e_7},\ y \to \frac{1+\frac{e_7}{h_1}x}{1+\frac{x}{e_3}} y\}$

$\to$ the relation $\boxed{s_2^2 = \mathrm{id}}$:

$$s_2(\frac{h_1}{e_7}) = \frac{h_1}{s_2(e_7)} = \frac{h_1}{h_1/e_3} = e_3.$$

$$s_2^2(e_3) = s_2\Big(s_2(e_3)\Big) = s_2\Big(\frac{h_1}{e_7}\Big) = e_3.$$

$$s_2^2(y) = s_2\Big(\frac{1+\frac{e_7}{h_1}x}{1+\frac{x}{e_3}} y\Big) = \frac{1+\frac{1}{e_3}x}{1+\frac{e_7}{h_1}x} s_2(y) = y.$$

• Check of $\boxed{s_2 s_3 s_2(y) = s_3 s_2 s_3(y)}$.

By definition

$$s_2(y) = (1 + \frac{e_7}{h_1}x)(1 + \frac{x}{e_3})^{-1}y,$$

$$s_3 = \{e_1 \to \frac{h_2}{e_5},\ e_5 \to \frac{h_2}{e_1},\ h_1 \to \frac{h_1 h_2}{e_1 e_5},\ x \to x\frac{1 + \frac{h_2}{e_1}y}{1 + e_5 y}\},$$

we have

$$s_3(s_2(y)) = s_3\Big((1 + \frac{e_7}{h_1}x)(1 + \frac{x}{e_3})^{-1}y\Big)$$

$$= \Big(1 + \frac{e_1 e_5 e_7}{h_1 h_2}x\frac{1 + \frac{h_2}{e_1}y}{1 + e_5 y}\Big)\Big(1 + \frac{1}{e_3}x\frac{1 + \frac{h_2}{e_1}y}{1 + e_5 y}\Big)^{-1}y.$$

$$s_3 s_2(y) = \Big(1 + \frac{e_1 e_5 e_7}{h_1 h_2} x \frac{1 + \frac{h_2}{e_1} y}{1 + e_5 y}\Big)\Big(1 + \frac{1}{e_3} x \frac{1 + \frac{h_2}{e_1} y}{1 + e_5 y}\Big)^{-1} y.$$

Using $\boxed{AB^{-1} = ACC^{-1}B^{-1} = (AC)(BC)^{-1}}$, we have

$$= \Big(1 + e_5 y + \frac{e_1 e_5 e_7}{h_1 h_2} x (1 + \frac{h_2}{e_1} y)\Big)\Big(1 + e_5 y + \frac{1}{e_3} x (1 + y \frac{h_2}{e_1})\Big)^{-1} y$$

$$= \Big(1 + \frac{e_1 e_5 e_7}{h_1 h_2} x + e_5 (1 + \frac{e_7}{h_1} x) y\Big)\Big(1 + \frac{1}{e_3} x + (e_5 + \frac{h_2}{e_1 e_3} x) y\Big)^{-1} y.$$

The last expression is s_2-invariant, hence

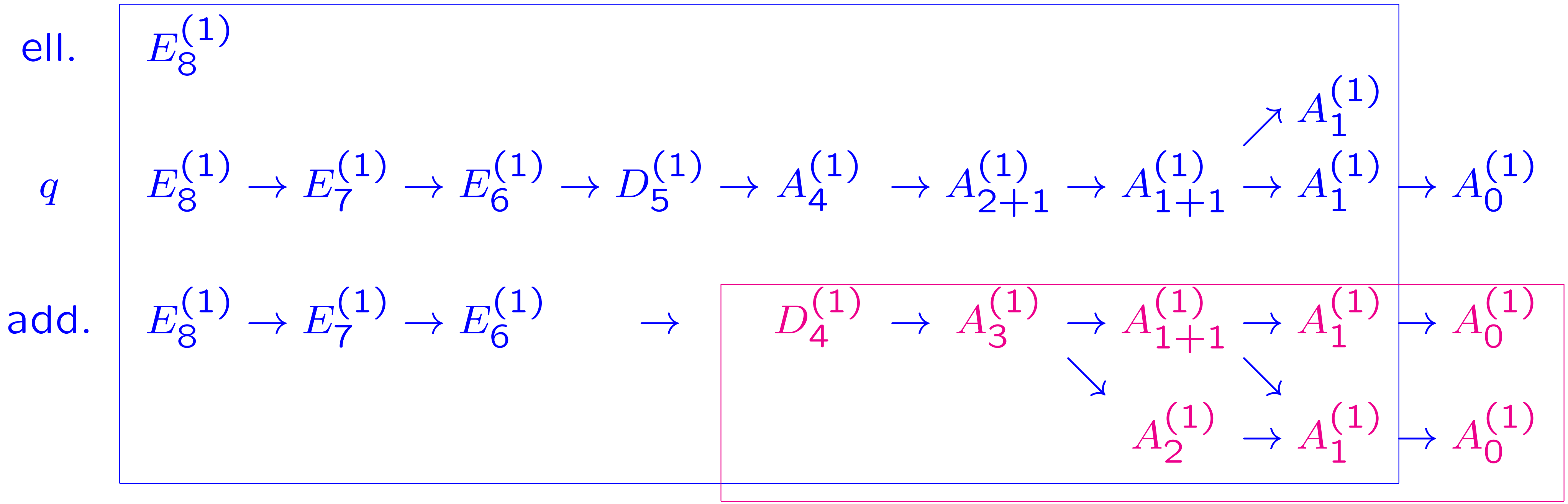
$$\boxed{s_2 s_3 s_2(y) = s_3 s_2(y) = s_3 s_2 s_3(y)}.$$

- **We don’t need any commutation relations between x and y.**

$\rightarrow$ **The Weyl group relations hold true also when x, y are non-commutative** [Hasegawa(2007)].

- $W(D_5^{(1)})$ gives the q-P_{VI} and its symmetry.

(commuative case [Jimbo-Sakai(1996)], quantum case [Hasegawa(2007)])



- Our target is the quantum version of the q-difference $E_8^{(1)}$ case.

2. The representation of $W(E_8^{(1)})$

▲ **Various root systems** can be realized on

$$X = \mathrm{Bl}_8(\mathbb{P}^1 \times \mathbb{P}^1).$$

- **e.g.** 2+2+2+2 points on 4 lines:

$$-\mathcal{K}_X = \delta_0 + \delta_1 + \delta_2 + \delta_3.$$

$$\delta_1 = \boxed{H_1 - E_1 - E_2}, \quad \delta_2 = \boxed{H_2 - E_3 - E_4},$$

$$\delta_3 = \boxed{H_1 - E_5 - E_6}, \quad \delta_0 = \boxed{H_2 - E_7 - E_8}.$$

$\to$ Roots $R := \langle \delta_0, \delta_1, \delta_2, \delta_3 \rangle^{\perp} = D_5^{(1)}$. $\qquad (E_{ij} = E_i - E_j)$

$$\begin{array}{ccccccc} & & \boxed{E_{12}} & & \boxed{E_{34}} & & \\ & & | & & | & & \\ \boxed{E_{56}} & - & \boxed{H_2 - E_1 - E_5} & - & \boxed{H_1 - E_3 - E_7} & - & \boxed{E_{78}} \end{array}$$

$\to$ **affine Weyl group** $W(D_5^{(1)})$.

• **e.g.** 4+2+2 points on a curve and 2 lines:

$$-\mathcal{K}_X = \boxed{H_1 + H_2 - E_1 - \cdots - E_4} + \boxed{H_1 - E_5 - E_6} + \boxed{H_2 - E_7 - E_8}.$$

→ **affine Weyl group $W(E_6^{(1)})$.**

• **e.g.** 8 points on an elliptic curve (smooth/nodal/cusp.):

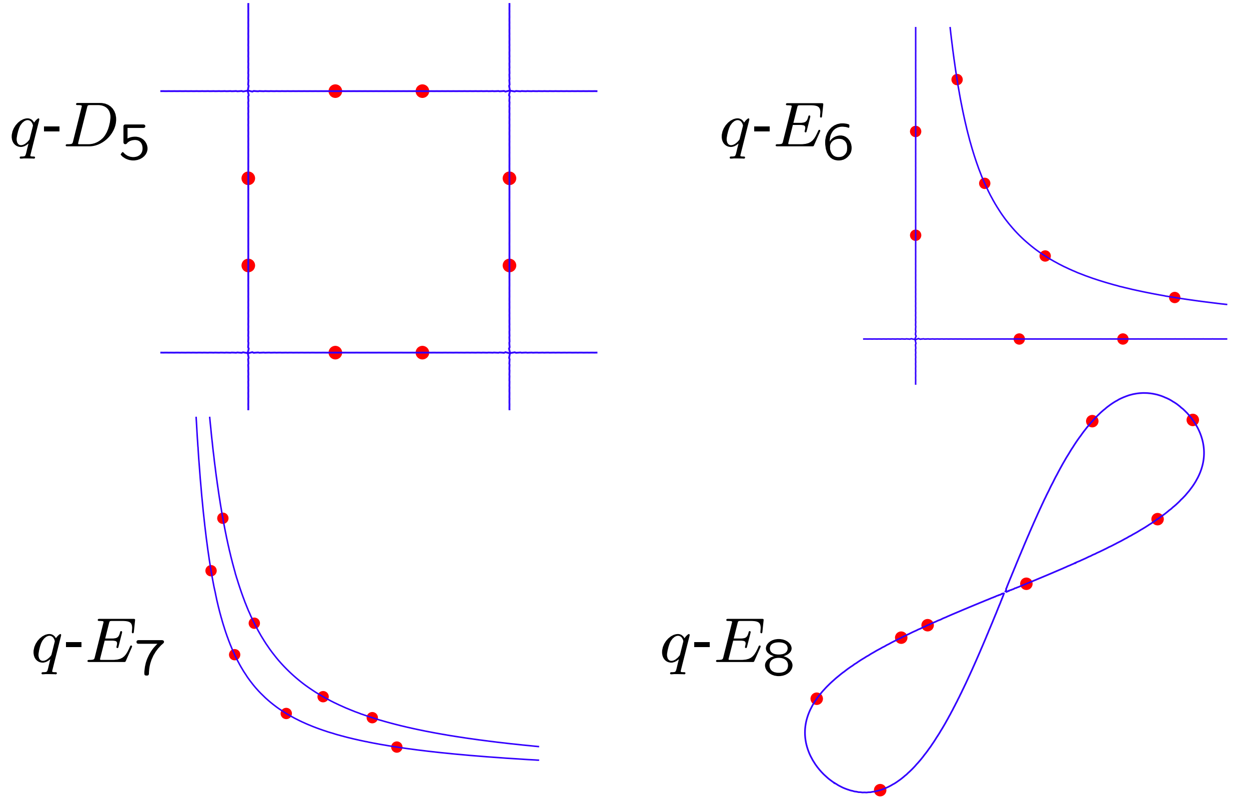
$$-\mathcal{K}_X = \delta = \boxed{2H_1 + 2H_2 - E_1 - \cdots - E_8}.$$

• Roots $R = \langle \delta \rangle^{\perp} = E_8^{(1)}$:

$$\begin{array}{ccccccccccccc} & & & & \boxed{E_{12}} & & & & & & \\ & & & & | & & & & & & \\ \boxed{H_1 - H_2} & - & \boxed{H_2 - E_1 - E_2} & - & \boxed{E_{23}} & - & \boxed{E_{34}} & - & \cdots & - & \boxed{E_{78}} \end{array}$$

→ **affine Weyl group $W(E_8^{(1)})$.**

▲ Configurations for $E_n^{(1)}$:

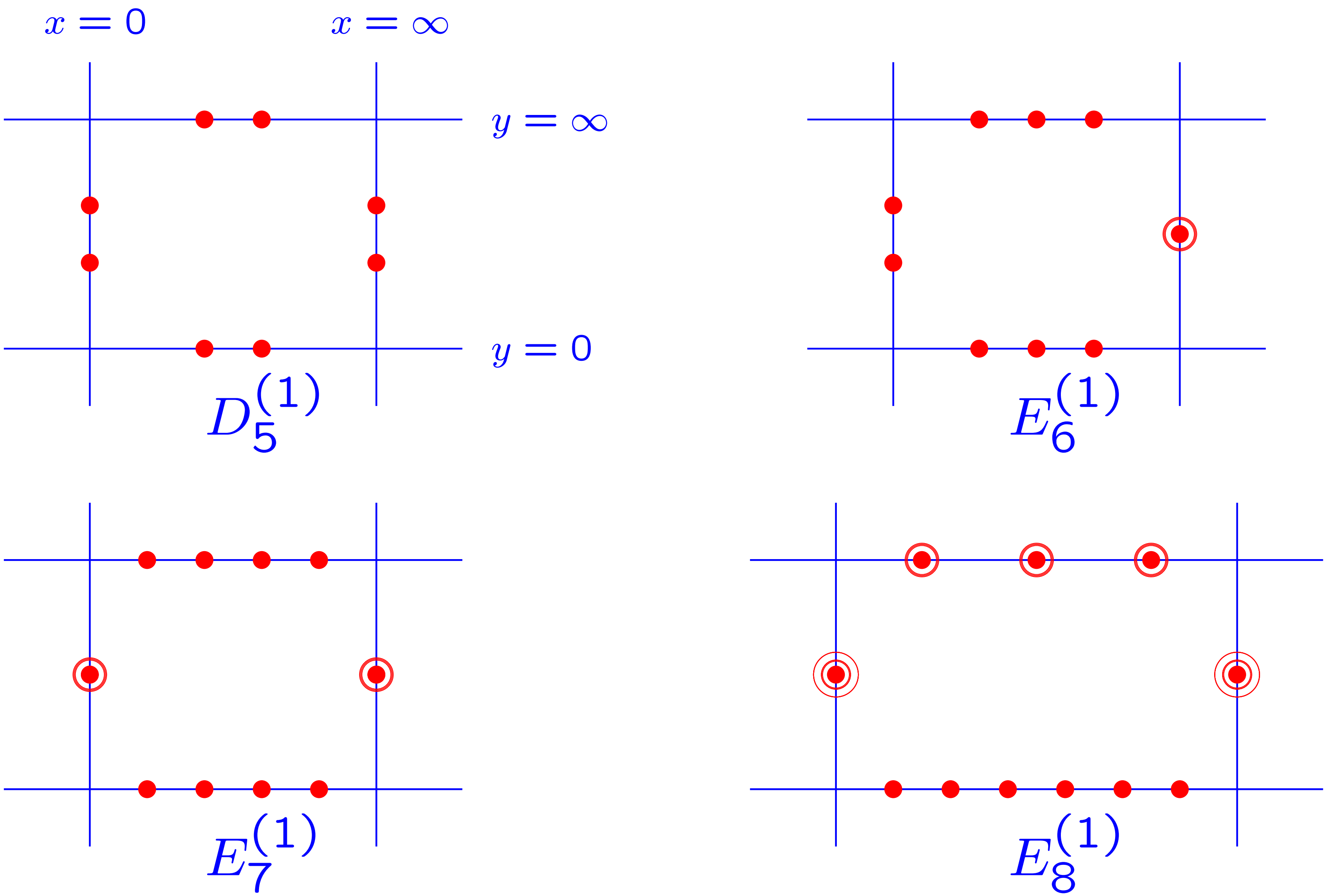


• For $D_5^{(1)}$, we have $\omega = \frac{dx \wedge dy}{xy} \to$ Poisson bracket $\{x, y\} = xy$.

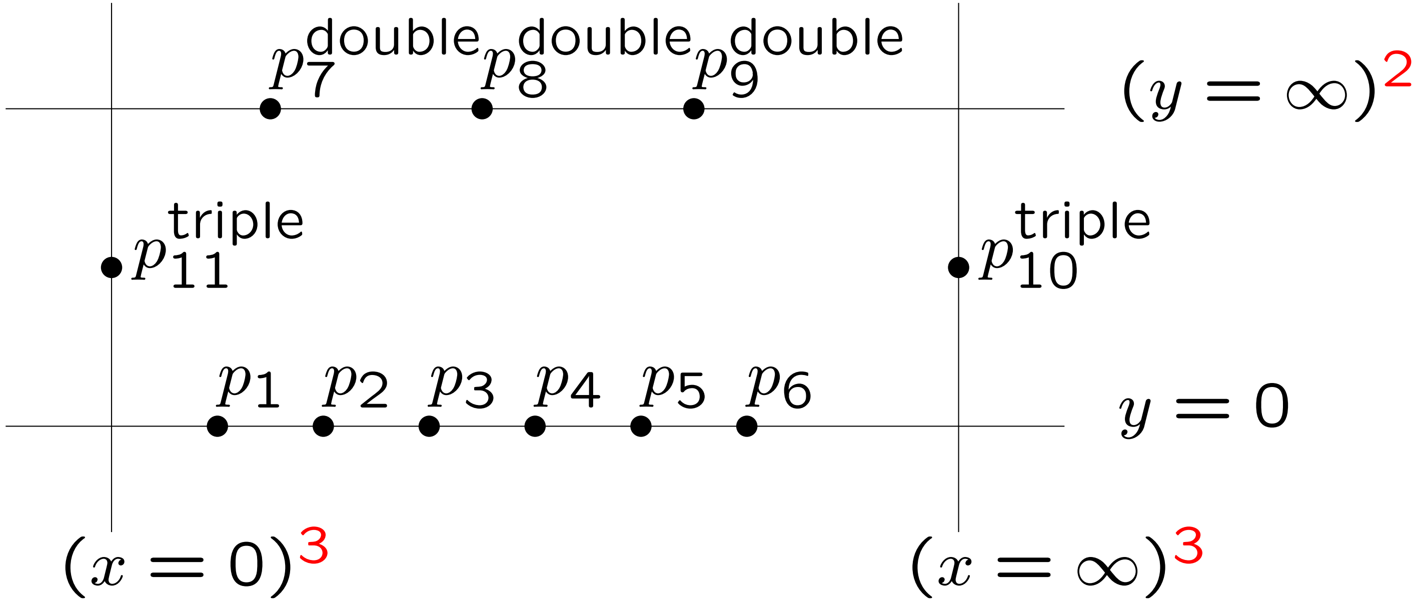
But for $E_n^{(1)} \to$ quantization is not so easy.

e.g. $\{x, y\} = xy(xy - 1)$, (for $E_6^{(1)}$).

▲ **Another realization.** $\mathrm{Bl}_k(\mathbb{P}^1 \times \mathbb{P}^1)$, $k > 8$ (degenerate config.)



- $E_8^{(1)}$:



- $-\mathcal{K}_X$ is of high degree but $g=1$ due to the **singularities**.

$$-\mathcal{K}_X = 6H_1+3H_2-\sum_{i=1}^{6} E_i-2\sum_{i=7}^{8} E_i-3\sum_{i=10}^{11} E_i$$

$$= \boxed{H_2-\sum_{i=1}^{6} E_i}+2\boxed{H_2-\sum_{i=7}^{8} E_i}+3\boxed{H_1-E_{10}}+3\boxed{H_1-E_{11}}.$$

• **Thm.** Let $k=\mathbb{C}(h_1,h_2,e_1,\ldots,e_{11})$. On a skew field $K=k(x,y)$, we have a birational representation of $W(E_8^{(1)})$.

$$s_0=\{e_{10}\to\frac{h_2}{e_{11}},\ e_{11}\to\frac{h_2}{e_{10}},\ h_1\to\frac{h_1h_2}{e_{10}e_{11}},\ x\to x\frac{1+y\frac{h_2}{e_{10}}}{1+ye_{11}}\},$$

$$s_1=\{e_8\leftrightarrow e_9\},\quad s_2=\{e_7\leftrightarrow e_8\},$$

$$s_3=\{e_1\to\frac{h_1}{e_7},\ e_7\to\frac{h_1}{e_1},\ h_2\to\frac{h_1h_2}{e_1e_7},\ y\to\frac{1+x\frac{e_7}{h_1}}{1+\frac{x}{e_1}}y\},$$

$$s_4=\{e_1\leftrightarrow e_2\},\quad s_5=\{e_2\leftrightarrow e_3\},\quad s_6=\{e_3\leftrightarrow e_4\},$$

$$s_7=\{e_4\leftrightarrow e_5\},\quad s_8=\{e_5\leftrightarrow e_6\}.$$

• In commutative case, this rep was known (e.g. [Tsuda (2006)][Tsuda-Takenawa(2009)]).

• Similar to the $D_5^{(1)}$ case, the actions give a representation also when **x, y are non-commutative**.

• To apply the rep to Painlevé equation, we want to compute the action of **translations**.

For $E_8^{(1)}$ case, we have (2×)120 directions. Each of them is given by **58** simple reflections $\to$ **too big!**

• In commutative case, we have the following factorization

$$w(x) = \frac{A}{B}, \quad w(y) = \frac{C_1 C_2 \cdots C_6}{D_1 D_2 D_3}, \quad w \in W(E_8^{(1)}).$$

Here A, B, C_i, D_i are some **polynomials** in x, y. They are complicated for general w, but have a simple geometric characterization. [Kajiwara et.al(2003)] [Tsuda(2006)][Tsuda-Takenawa(2009)]

• To study these polynomials, a **lift of the rep including tau-variables** is essential. Its quantization is our next problem.

3. Lifting the representation including τ variables

- **τ-variables.** In addition to h_i, e_i, x, y, we introduce new variables

$$\sigma_1, \sigma_2, \tau_1, \ldots, \tau_{11}.$$

- The following q-commutation relations are crucial

$$yx = qxy$$

$$\sigma_i h_j = q^{H_i.H_j} h_j \sigma_i, \quad \tau_i e_j = q^{E_i.E_j} e_j \tau_i.$$

- **Note.** The parameters $\{h_i, e_i\}$ and the τ-variables $\{\sigma_i, \tau_i\}$ are **non-commutative**. This important idea is borrowed from the formulation by Kuroki [arXiv:1206.3419(math-QA)], where he found

$$\text{simple coroots} : \alpha_i^\vee \underset{\text{canonical conjugate}}{\longleftrightarrow} \tau\text{- variables} : \tau_i.$$

• **Thm.** One can extend the quantum rep $W(E_8^{(1)})$ on $\mathbb{C}_{\text{skew}}(h_i, e_i, x, y)$ to $\mathbb{C}_{\text{skew}}(h_i, e_i, x, y, \sigma_i, \tau_i)$ as algebra auto

$$s_0 = \{\tau_{10} \to (1+ye_{11})\frac{\sigma_2}{\tau_{11}},\ \tau_{11} \to \frac{\sigma_2}{\tau_{10}}(1+y\frac{h_2}{e_{10}}),\ \sigma_1 \to (1+ye_{11})\frac{\sigma_1\sigma_2}{\tau_{10}\tau_{11}}\},$$

$$s_1 = \{\tau_8 \leftrightarrow \tau_9\}, \quad s_2 = \{\tau_7 \leftrightarrow \tau_8\},$$

$$s_3 = \{\tau_1 \to (1+x\frac{e_7}{h_1})\frac{\sigma_1}{\tau_7},\ \tau_7 \to \frac{\sigma_1}{\tau_1}(1+\frac{x}{e_1}),\ \sigma_2 \to \frac{\sigma_1\sigma_2}{\tau_1\tau_7}(1+\frac{x}{e_1})\},$$

$$s_4 = \{\tau_1 \leftrightarrow \tau_2\}, \quad s_5 = \{\tau_2 \leftrightarrow \tau_3\}, \quad s_6 = \{\tau_3 \leftrightarrow \tau_4\},$$

$$s_7 = \{\tau_4 \leftrightarrow \tau_5\}, \quad s_8 = \{\tau_5 \leftrightarrow \tau_6\}.$$

(The actions on $\{h_i, e_i, x, y\}$ are the same as before.)

• When $x = y = 0$, the actions on $\{\sigma_i, \tau_i\}$ are just a copy of the actions on $\{h_i, e_i\}$.

e.g. For $w = s_0 s_3 s_4 s_0 s_2 s_3 s_2 s_1 s_0 s_2 s_4 s_3$, we have

$$w(e_{11}) = \frac{h_1^2 h_2^2}{e_1 e_2 e_7 e_8 e_{10}^2 e_{11}}, \qquad \boxed{w(\tau_{11}) = F(x,y)\frac{\sigma_1^2\sigma_2^2}{\tau_1\tau_2\tau_7\tau_8\tau_{10}^2\tau_{11}}},$$

$$\begin{aligned} F(x,y) &= (1+\tfrac{x}{e_1 q})(1+\tfrac{x}{e_2 q}) + (* + *x + *x^2)y \\ &\quad + * (1+\tfrac{e_7}{h_1}x)(1+\tfrac{e_8}{h_1}x)y^2 \\ &= (1+e_{11}y)(1+w(e_{11})y) + x(1+\tfrac{h_2}{e_{10}}y)(* + *y) \\ &\quad + * x^2(1+\tfrac{h_2}{e_{10}}y)(1+\tfrac{qh_2}{e_{10}}y). \end{aligned}$$

- **Regularity.** For any $w \in W(E_8^{(1)})$, we see

$$w(\tau_i) = F_{i,w}(x,y) \times (\text{monomial of } \{\sigma_j, \tau_j\}),$$

where $F_{i,w}(x,y)$ is a non-commutative **polynomial** in x, y (cf. "Laurent phenomena", "singularity confinement"). We will clarify the reason.

4. F-polynomials and the quantum curve

• When $q = 1$, the regularity of $F(x,y)$ is known as follows.

• The polynomial F can be determined by the data (d_i, m_i) through the linear system:

$$(F=0) \in \left| \lambda := \sum_{i=1}^{2} d_i H_i - \sum_{i=1}^{11} m_i E_i \in \mathrm{Pic}(X) \right|.$$

• In particular, for $\lambda \in$ EX (**exceptional class**) = $W(E_8^{(1)})$ orbit of $\{E_i\}$, the corresponding curves $F(x,y) = 0$ are rigid and $g = 0$.

These polynomials give the factors of the rational expressions of $w(x), w(y)$.

• We will formulate the analog of these properties for $q \neq 1$.

Where $F(x,y)$ is non-commutative $(yx = qxy)$, i.e. a q-difference operator.

• **Non-logarithmic singularity.**([Carmichael], [Birkhoff], [Adams])

Consider a q-difference equation $D\psi(x) = 0$ where

$$\boxed{D = A_0(y) + xA_1(y) + x^2 A_2(y) + \cdots}. \quad (yx = qxy)$$

• **Exponents**: $A_0(q^\rho) = 0 \to {}^\exists\psi(x) = x^\rho(1 + c_1 x + \cdots)$.

• **Resonances** of exponents ($\rho' - \rho \in \mathbb{Z}$) generically bring log-terms to $\psi(x)$. However, in some special **"non-logarithmic"** cases, one can still have solutions without log-terms.

e.g.

$$A_0 \propto (y - q^\rho)(y - q^{\rho+1})(y - q^{\rho+2}),$$

$$A_1 \propto (y - q^\rho)(y - q^{\rho+1}), \quad A_2 \propto (y - q^\rho).$$

• Our $F(x, y)$ operators have many resonances, but **they all are non-logarithmic!**

• **Def.** For a data $\lambda = (d_i, m_i)$, we define a q-difference operator $F_\lambda(x,y)$ so that the following two expressions are consistent:

$$
\begin{aligned}
F_\lambda &= \sum_{i=0}^{d_1} x^i \prod_{t=i}^{m_{11}-1} (1+q^t e_{11} y) \prod_{t=d_1-m_{10}}^{i-1} \left(1+q^t \frac{h_2}{e_{10}} y\right) U_i(y), \\
&= \sum_{i=0}^{d_2} \prod_{k=1}^{6} \prod_{t=i-m_k}^{-1} \left(1+q^t \frac{1}{e_k} x\right) \prod_{k=7}^{9} \prod_{t=0}^{i-d_2+m_k-1} \left(1+q^t \frac{e_k}{h_1} x\right) V_i(x)\, y^i,
\end{aligned}
$$

Here U_i, V_i are polynomials: $\deg_y(U_i) = d_2-(i-d_1+m_{10})_+-(m_{11}-i)_+$, $\deg_x(V_i) = d_1-\sum_{k=1}^{6}(m_k-i)_+-\sum_{k=7}^{9}(i-d_2+m_k)_+$, $(x)_+ = \max(x,0)$.

• The 1st [or 2nd] line specifies the **non-logarithmic** singularities around $x = 0, \infty$ [or $y = 0, \infty$].

(In the latter, x is viewed as a q-shift operator: $x\psi(y) = \psi(q^{-1}y)$.)

• **Main Thm.** For $\lambda = \sum_{i=1}^{2} d_i H_i - \sum_{i=1}^{11} m_i E_i = w(E_i) \in$ EX, the quantum polynomial F_λ is unique (under the normalization $F_\lambda(0,0) = 1$). Moreover, it coincides with $F_{i,w}$ generated by the Weyl group action:

$$F_{i,w}(x, y) = F_\lambda(x, y).$$

This shows the regularity of $F_{i,w}$ and its geometric characterization.

• **A key fact for the proof**: The non-logarithmic property of $F_{i,w}$ is preserved under the Weyl group actions.

This fact is proved using a realization of the Weyl group actions as gauge transformations (gauge factor = q-dilogarithm).

• **Bilinear equations.** Consider an **infinite system of bilinear equations** generated by the seed equations ($1 \le i \le 6$ and $7 \le j \le 9$)

$$
\begin{aligned}
&\tau(e_{10})\tau(\tfrac{h_2}{e_{10}}) = \tfrac{h_2}{e_{10}}\tau(\tfrac{h_2}{e_i})\tau(e_i) + \tau(\tfrac{h_2}{e_j})\tau(e_j),\\
&\tau(\tfrac{h_2}{e_{11}})\tau(e_{11}) = e_{11}\tau(\tfrac{h_2}{e_i})\tau(e_i) + \tau(\tfrac{h_2}{e_j})\tau(e_j),\\
&\tau(e_i)\tau(\tfrac{h_1}{e_i}) = \tfrac{1}{e_i}\tau(\tfrac{h_1}{e_{11}})\tau(e_{11}) + \tau(\tfrac{h_1}{e_{10}})\tau(e_{10}),\\
&\tau(\tfrac{h_1}{e_j})\tau(e_j) = \tfrac{e_j}{h_1}\tau(\tfrac{h_1}{e_{11}})\tau(e_{11}) + \tau(\tfrac{h_1}{e_{10}})\tau(e_{10}),\\
&\tau(\tfrac{h_2}{e_1})\tau(e_1) = \ldots = \tau(\tfrac{h_2}{e_6})\tau(e_6),\\
&\tau(\tfrac{h_2}{e_7})\tau(e_7) = \ldots = \tau(\tfrac{h_2}{e_9})\tau(e_9),
\end{aligned}
$$

and their $W(E_8^{(1)})$ transformations (obtained by $w(\tau(\lambda)) = \tau(w\cdot\lambda)$).

• **Thm.** The **overdetermined system** given above is consistent and has a solution given by $\tau(\lambda) = F_\lambda(x,y)\tau^\lambda$.

$\rightarrow$ a quantum Plücker embedding of the Okamoto space X.

▲ Application to the quantum mirror curve.

- For generic parameters (h_i, e_i), the curve C in the linear system

$$|-\mathcal{K}_X| = \left|6H_1 + 3H_2 - \sum_{i=1}^{6} E_i - 2\sum_{i=7}^{9} E_i - 3\sum_{i=10}^{11} E_i\right|$$

is unique $g(x,y) = x^3 y = 0$ $(X_0^3 X_1^3 Y_0^2 Y_1 = 0)$.

- If the parameters are special: $p := \dfrac{h_1^6 h_2^3}{(e_1 \cdots e_6)(e_7 e_8 e_9)^2 (e_{10} e_{11})^3} = 1$,

$\to$ The curve $C \in |-\mathcal{K}_X|$ form **a pencil** $\boxed{\lambda f(x,y) + \mu g(x,y) = 0}$.

$\to$ The Painlevé equation reduces to an autonomous integrable system where the pencil gives **the algebraic integral**.

- The curve is the **quantum curve for E_8** [Moriyama (2020)].

It is also related to **Ruijsenaars - van Diejen operator** [Takemura (2018)] , [Noumi-Ruijsenaars-Y (2020)], [Chen-Haghighat-Kim-Sperling-Wang (2021)].

- From $W(E_8^{(1)})$ symmetry, one can determine the curve explicitly.

$$\boxed{\lambda\left(\sum_{i=0}^{3} C_i(x)y^i\right) + \mu x^3 y = 0}.$$

$$C_3(x) = q^3 e_{11}^3 \prod_{i=7}^{9} \left(1+\frac{e_i}{h_1}x\right)\left(1+q\frac{e_i}{h_1}x\right),$$

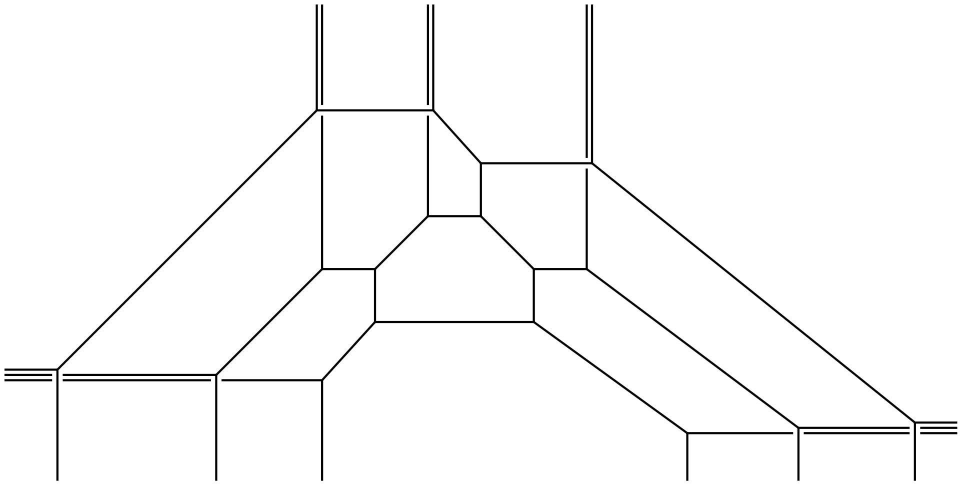
$$C_2(x) = q e_{11}^2 \prod_{i=7}^{9} \left(1+\frac{e_i}{h_1}x\right)\{[3]_q + qxA_{-1} + q\kappa A_1 x^2 + [3]_q \kappa x^3\},$$

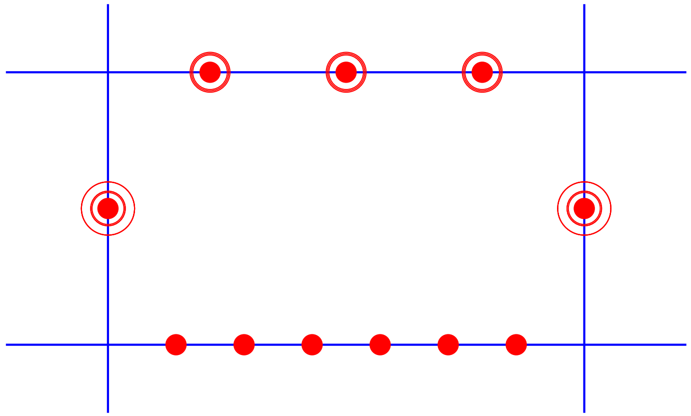
$$C_1(x) = e_{11}\{[3]_q + [2]_q A_{-1}x + (\kappa A_1 + A_{-2})x^2 + \frac{\kappa}{q}(\kappa A_2 + A_{-1})x^4 + \frac{[2]_q \kappa^2 A_1}{q^2}x^5 + \frac{[3]_q \kappa^2}{q^3}x^6\}, \qquad C_0(x) = \prod_{i=1}^{6}\left(1+\frac{1}{qe_i}x\right),$$

$$[k]_q = \frac{1-q^k}{1-q}, \quad A_{\pm 1} = \sum_{i=1}^{9} a_i^{\pm 1}, \quad A_{\pm 2} = \sum_{1\le i<j\le 9} (a_i a_j)^{\pm 1},$$

$$a_i = e_i \ (1 \le i \le 6), \quad a_i = \frac{h_1}{e_i} \ (7 \le i \le 9) \quad \kappa = \frac{e_7 e_8 e_9 e_{10} e_{11}}{h_1^2 h_2}.$$

• The **quantum curve** for $E_8^{(1)}$ was first obtained by S.Moriyama (2020) [arXix:2007.05148] as a quantization of a commutative case [Kim-Yagi (2015)].





• The spectral determinant (ST) of the **quantum curve** can be computed exactly and it gives the full partition function of the topological string (TS) on an open Calabi-Yau $=-\mathcal{K}_X$ over $X \simeq \mathrm{Bl}_9(\mathbb{P}^2)$.

(Known as the **TS/ST duality** [Grassi-Hatsuda-Marino(arXiv:1401.3382)] - one of the very sophisticated version of the mirror symmetry-. Though this is confirmed for various examples, many problems, in particular for $E_n^{(1)}$ cases, are waiting for the challenges.)

▲ Summary of the results

- We contracted a **quantum** birational rep of affine Weyl group $W(E_8^{(1)})$.
- A **lift** of the rep including the **tau variables** is also obtained.
- **Regularity** of the quantum polynomials F and their geometric characterization are proved.
- The quantum **mirror curve** of type q-$E_8^{(1)}$ is rederived from its symmetry.

▲ Future studies

- There are many problems to be clarified.

Thank you for your attention!