Algebraic relations between solutions of a differential equation.

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- $P_I : y'' = 6y^2 + x$

Theorem (K. Nishioka - 2004)

Let $y_1, \dots y_n$ be distinct solutions of P_I. Then

$$\text{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, y_1', \dots y_n, y_n') = 2n$$

Problem : To give $J \in \{II, \dots, VI\}$ and parameters α such that for n distinct solutions of $P_J(\alpha)$:

$$\text{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, y_1', \dots y_n, y_n') = \sum \text{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, y_i')$$

- $P_{II}(\alpha) : y'' = 2y^3 + xy + \alpha$
- $P_{III}(\alpha) : y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{\alpha_1 y^2 + \alpha_2}{x} + \alpha_3 y^3 + \frac{\alpha_4}{y}$
- $\dots P_{IV}(\alpha), P_V(\alpha), P_{VI}(\alpha)$

Theorem (J. Nagloo & A. Pillay - 2017)

Let $J \in \{II, \dots VI\}$ and α algebraically independent over $\mathbb{Q}$.
If $y_1, \dots y_n$ are solutions of $P_J(\alpha)$ such that

$$\mathrm{deg.tr.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, y_1', \dots y_n, y_n') < 2n$$

then

- $\exists i, \mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, y_i') < 2$
- $\exists i < j, \mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, y_i', y_j, y_j') = 2$

The proof uses

1. The classification of non algebraic classical solutions (Nishioka, Umemura, Okamoto-Noumi, Watanabe ...)
2. The trichotomy theorem in DCF_0 (Hrushovski-Sokolovic)
3. Elimination of quantifiers in $DCF_0 + (1)$.

Then using the classification of algebraic solutions, they can partially solve the problem

Theorem (J. Nagloo & A. Pillay - 2014)

Let $J \in \{II, \ldots V\}$ and α algebraically independent over $\mathbb{Q}$. if $y_1, \ldots y_n$ are distincts solutions of $P_J(\alpha)$ then

$$\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, y_1', \ldots y_n, y_n') = 2n$$

Variation of Nagloo-Pillay Theorem

$$(X): \quad \begin{aligned} y_1^{(m-1)} &= f(x, y_1, \ldots, y_1^{(m-2)}), \\ &\vdots \\ y_n^{(m-1)} &= f(x, y_n, \ldots, y_n^{(m-2)}). \end{aligned}$$

If Malgrange pseudogroup of X

= Galois groupoid = Umemura's infinitesimal group

is big enought = infinite dimensional, simple and primitive

then

if a solution $y(x)$ of$X^{(n)}$ satisfies

$$\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_1, \ldots, y_1^{(m-2)}, \ldots, y_n, \ldots, y_n^{(m-2)}) < n(m-1)$$

then

- $\exists i$, $\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, \ldots, y_i^{(m-2)}) < m-1$.
- $\exists i < j$, $\mathrm{tr.deg.}_{\mathbb{C}(x)}\mathbb{C}(x)(y_i, \ldots, y_i^{(m-2)}, y_j, \ldots, y_j^{(m-2)}) = m-1$

M an alg. variety over $\mathbb{C}$, $\dim M = m$, $\mathbb{C}(M)$ the field of rational functions.
X a rational vector field on M.

Differential invariants of X

- $\partial_1, \dots \partial_m$ some symbols and $\mathbb{C}(M)_\infty$ the ∂-differential field generated by $\mathbb{C}(M)$.
- X_∞ the extension of X commuting with ∂s
- $\mathrm{Inv}(X) = \{H \in \mathbb{C}(M)_\infty : X_\infty \cdot H = 0\}$

Example $M = \mathbb{A}^m$, $\mathbb{C}(M) = \mathbb{C}(x_1, \dots x_m)$, $X = \sum a_i(x) \frac{\partial}{\partial x_i}$

- $\mathbb{C}(M)_\infty = \mathbb{C}(x_{i,\alpha} | i = 1, \dots m; \alpha \in \mathbb{N}^m)$; $\partial_j(x_{i,\alpha}) = x_{i,\alpha+1_j}$
- $X_\infty = \sum \partial^\alpha(a_i) \frac{\partial}{\partial x_{i,\alpha}}$
- $L_X(\sum w_i(x) dx_i) = 0$ if and only if $\forall j,\ X_\infty \cdot \left(\sum w_i(x) x_{i,1_j}\right) = 0$.

For $\varphi : \mathcal{U} \to \mathcal{V}$ a bihiolomorphism between open subsets of M, $\varphi^*_\infty : Mer(\mathcal{V})_\infty \to Mer(\mathcal{U})_\infty$ is the morphism extending φ^* and commuting with ∂s.

Definition

$$\mathrm{Mal}(X) = \{\varphi \mid \forall H \in \mathrm{Inv}(X)\ \varphi^*_\infty(H) = H\}$$

Example If $L_X\omega = 0$ then $\forall\varphi \in \mathrm{Mal}(X),\ \varphi^*\omega = \omega$.

Example $X = \frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + a_3(x)\frac{\partial}{\partial x_3}$ and $\exists\theta$, 2-form such that $d\theta = 0,\ i_X\theta = 0$.

$$\mathrm{Mal}(X) \subset \{\varphi \mid \varphi^*X = X, \varphi^* dx_1 = dx_1, \varphi^*\theta = \theta\}$$

In local coordinates x_1, y, z such that $X = \frac{\partial}{\partial x_1}$ and $\theta = dy \wedge dz$,

$$\varphi(x_1, y, z) = (x_1 + c, f(y, z), g(y, z)) \text{ with } \frac{\partial(f,g)}{\partial(y,z)} = 1.$$

Painlevé equations $P_J(\alpha)$ are examples :

- a vector field $X_J(\alpha) = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + (\ldots\ldots) \frac{\partial}{\partial x_3}$
- a 2-form $\theta_J(\alpha)$ t.q. $i_{X_J(\alpha)} \theta_J(\alpha) = 0$

Theorem

$$\mathrm{Mal}(X_J(\alpha)) = \{\varphi \mid \varphi^* X_J(\alpha) = X_J(\alpha), \varphi^* dx_1 = dx_1, \varphi^* \theta_J(\alpha) = \theta_J(\alpha)\}$$

- *$J = I$*
- *$J = VI$ except for Picard parameters (Cantat-Loray)*
- *$J = II$, $\alpha \in \mathbb{N}$ (+ Weil)*
- *Any J, α general (Davy)*

• $\pi : M \to \mathbb{A}^1$ avec $d\pi(X) = \frac{\partial}{\partial x_1}$ and θ a closed $m-1$-form with $i_X\theta = 0$.

• $M^{(n)} = M \underset{\mathbb{A}^1}{\times} M^{(n-1)}$ and $X^{(n)}$ the sum of n copies de X sur $M^{(n)}$.

Theorem

Assume $\mathrm{Mal}(X) = \{\varphi \mid \varphi^*X = X, \varphi^* dx_1 = dx_1, \varphi^*\theta = \theta\}$

If $V \subsetneq M^{(n)}$ *is the Zariski closure of a trajectory of* $X^{(n)}$ *then*

- $\exists i$, $pr_i(V) \subset M$ *has dimension* $< m$
- $\exists i < j$, $pr_{i,j}(V) \subset M \underset{\mathbb{A}^1}{\times} M$ *has dimension* m.

$\mathrm{Mal}^0(X) = \mathrm{Mal}(X) \cap \{\varphi \mid \pi \circ \varphi = \pi\}$ is simple, primitive and infinite dimensional.

Induction on n and assume that projections $pr_1 : V \to M$ et $pr_{2,\dots n} : V \to M^{(n-1)}$ are dominant.

Theorem

If $\rho : M \dashrightarrow N$ is rational, dominant and $d\rho(X) = Y$ then ρ induced a dominant morphism $\rho_* : \mathrm{Mal}(X) \to \mathrm{Mal}(Y)$.

One gets dominant projections $(pr_i)_* : \mathrm{Mal}(X^{(n)}) \to \mathrm{Mal}(X)$

Lemma

$\mathrm{Mal}(X^{(n)}) = \mathrm{Mal}(X)^{(n)}$

- As $\mathrm{Mal}^0(X)$ is simple and projections are onto, il is enough to prove it for $n = 2$.
- If the inclusion is strict then one can assume $\mathrm{Mal}^0(X^{(2)})$ is the diagonal embedding of $\mathrm{Mal}^0(X)$ in $\mathrm{Mal}^0(X)^{(2)}$.
- Lie, Cartan : The diagonal embedding of a Lie pseudogroup is a Lie pseudogroup if and only if it is finite dimensional.

Lemma

$pr_{2,\ldots,n} : V \to M^{(n-1)}$ is generically finite.

Fibers give a finite dimensional family of subvarieties of dimension $< m-1$ in M included in $x_1 = cste$ and invariant under $\mathrm{Mal}(X)$ ...
This pseudogroup acts transitively on germs of curves.

Lemma

If $\mathcal{F}$ is a codimension m $X^{(n)}$-invariant foliation on V then $\exists\ i > 1$, leaves are fibers of pr_i .

Same argument is used on $M^{(n-1)}$ to describe invariants foliations under the action of $\mathrm{Mal}(X^{(n-1)}) = \mathrm{Mal}(X)^{(n-1)}$.

Apply this to the foliation of V by fibers of pr_1, the lemma proves the theorem.

Thank you for your attention