Moduli varieties of twisted local systems

Florent Schaffhauser

Universidad de Los Andes (Bogotá)

Web-seminar on Painlevé equations and related topics
神戸 – Rennes, 7 July 2021

Contents / 目次

Differential equations with symmetry

- Linear O.D.E. on a Riemann surface $Y \to$ locally constant sheaves of vector spaces / groups.
- O.D.E. with symmetry $\to$ group $\mathfrak{S}$ acting on Y.
- (orbifold) covering map $p : Y \longrightarrow [Y/\mathfrak{S}]$
  $\to$ twisted local system on $X := Y/\mathfrak{S}$.

Questions

- What is a twisted local system?
  A torsor (=principal homogeneous space) under a local system of groups.
- How to classify them?
  Cohomology with coefficients in a local system of groups.
- What can we say about the geometry of the moduli space?
  Algebraic structure, topology, etc.

Moduli varieties

- Constructed by P. Boalch and D. Yamakawa (arXiv:1512:08091).
- They are complex Poisson varieties (including in the irregular case).
- The proof uses twisted quasi-Hamiltonian geometry.

Twisted local systems

- X: a topological space, complex manifold *etc.*
- $\mathscr{G} \longrightarrow X$: a group cover of X.
  - A cover $\mathscr{G} \longrightarrow X$.
  - A morphism $\mu : \mathscr{G} \times_X \mathscr{G} \longrightarrow \mathscr{G}$ and a section $e : X \longrightarrow \mathscr{G}$.
  - Axioms: μ is associative and e defines a unit element.
- A twisted local system is a torsor under $\mathscr{G}$:
  - A cover $\mathscr{V} \longrightarrow X$.
  - A morphism $\mathscr{V} \times_X \mathscr{G} \longrightarrow \mathscr{V}$ + axioms of a group action.
  - The canonical morphism

$$\begin{array}{ccc} \mathscr{V} \times_X \mathscr{G} & \longrightarrow & \mathscr{V} \times_X \mathscr{V} \\ (p, g) & \longmapsto & (p, p \cdot g) \end{array}$$

    is an isomorphism.

Group coverings - Constant groups

- G: a (complex reductive) Lie group, with discrete form $G^\sharp$. $\mathscr{G} = X \times G^\sharp$: the trivial group covering of X.
- Principal homogeneous $\mathscr{G}$-space = principal $G^\sharp$-bundle (also called a G-local system).
- Sheaves of sections = locally constant sheaves of groups.

Group coverings - Nonconstant groups (example)

- $p : (Y, \sigma) \longrightarrow \mathbb{C}\mathbf{P}^1$: a hyperelliptic curve.
- $\mathfrak{S} := [\langle \sigma \rangle] \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $G := \mathbf{GL}(n; \mathbb{C})$ via $\sigma : g \longrightarrow {}^t g^{-1}$.
- $\mathscr{G} := \left[(Y \times G^\sharp)/\mathfrak{S}\right] \to$ group covering of $X := [Y/\mathfrak{S}]$ $(\sim \mathbb{C}\mathbf{P}^1)$.

$\mathscr{G}$-torsors (on X) $\leftrightarrow$ anti-invariant local systems on Y.

Group coverings

- $\mathscr{G} \longrightarrow X$ gives rise to $\widetilde{\mathscr{G}} \longrightarrow \widetilde{X}$, equivariant group covering with $\pi_1 X$-action.

Proposition

Necessarily: $\widetilde{\mathscr{G}} \simeq \widetilde{X} \times \Gamma$ with Γ discrete, and $\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(\Gamma)$ a group morphism.

- Case of special interest: $\Gamma = G^\sharp$ (discrete form of a Lie group), and $\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(G)$ with finite image.

Holonomy representations

- $\mathscr{G}_\varphi^\sharp := \left[(\widetilde{X} \times G^\sharp)/\pi_1 X\right]$, where $\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(G)$.
  $\mathscr{V}$: a $\mathscr{G}_\varphi^\sharp$-torsor.

Proposition

The $\pi_1 X$-equivariant structure on $\widetilde{\mathscr{V}} \simeq \widetilde{X} \times G^\sharp$ is given by a crossed morphism $\varrho : \pi_1 X \longrightarrow G$.

- $\varrho(\sigma_1\sigma_2) = \varrho(\sigma_1)\, \varphi_{\sigma_1}\big(\varrho(\sigma_2)\big)$.
  $\varrho' \sim \varrho$ if $\exists g, \forall \sigma, \varrho'(\sigma) = g\varrho(\sigma)\varphi_\sigma(g^{-1})$.

Classification of $\mathscr{G}$-torsors

- $\varrho : \pi_1 X \longrightarrow G$ a crossed morphism.
- Define

$$\mathscr{V}_\varrho := \left[(\widetilde{X} \times G^\sharp)/\pi_X \right]$$

where $\sigma \cdot (\xi, h) = \big(\sigma \cdot \xi, \varrho(\sigma)\varphi_\sigma(g)\big)$.

Theorem

The map $\varrho \longmapsto \mathscr{V}_\varrho$ induces an isomorphism

$$H^1_\varphi(\pi_1 X; G) \simeq \left\{\varphi\text{-twisted } G\text{-local systems}\right\} \big/ \sim .$$

Crossed morphisms *vs* representations

- Given $\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(G)$, define $\widehat{G} := G \rtimes \mathrm{Im}\,\varphi$.
- There is a bijection $\varrho \longmapsto \widehat{\varrho} := (\varrho, \varphi)$ between
$$H^1_\varphi(\pi_1 X; G) = Z^1_\varphi(\pi_1 X; G)/G$$
and
$$\mathrm{Hom}_\varphi(\pi_1 X; \widehat{G})/G$$
- $\widehat{\varrho}$ is called the extended holonomy representation.

Equivariant picture

- Define $p : X_\varphi \longrightarrow X$ (cover) by $\pi_1 X_\varphi := \ker\varphi \subset \pi_1 X$. Then $p^*\mathscr{G}_\varphi^\sharp \simeq X_\varphi \times G^\sharp$.
- $H^1_\varphi(\pi_1 X; G)$ parameterises representations satisfying

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1 X_\varphi & \longrightarrow & \pi_1 X & \longrightarrow & \operatorname{Im}\varphi & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varrho|_{\pi_1 X_\varphi}} & & \downarrow{\scriptstyle \widehat{\varrho}} & & \| & & \\ 1 & \longrightarrow & G & \longrightarrow & G \rtimes \operatorname{Im}\varphi & \longrightarrow & \operatorname{Im}\varphi & \longrightarrow & 1 \end{array}$$

- φ-twisted G-local systems on X:

$$\left\{\mathscr{G}_\varphi^\sharp\text{-torsors}\right\} \leftrightarrow \left\{(\operatorname{Im}\varphi)\text{-equivariant } G\text{-local systems on } X_\varphi\right\}$$

Special holonomy

- Involution θ of G, acting trivially on Y.
- Set $X := [Y/\langle\theta\rangle]$. Then $\pi_1 X \simeq \pi_1 X \times \langle\theta\rangle$.

Proposition

There is a bijection

$$H^1_\varphi(\pi_1 X; G) \simeq \mathrm{Hom}(\pi_1 X; G^\theta)/G^\theta$$

- In particular, G^θ can be a real form of G.

Affine quotients

- G algebraic $\Rightarrow Z^1_\varphi(\pi_1 X; G)$ is an affine algebraic set.
- G reductive: $\exists$ a GIT quotient
$$\mathcal{M}_{\mathrm{B}}(X; \mathscr{G}_\varphi) := Z^1_\varphi(\pi_1 X; G)//G$$
- Points of $\mathcal{M}_{\mathrm{B}}(X; \mathscr{G}_\varphi)$ = closed G-orbits in $Z^1_\varphi(\pi_1 X; G)$.

Closed orbits

- GIT: $\mathcal{O}_1 \sim \mathcal{O}_2$ if $\overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2} \neq \emptyset$.
- Better understood in terms of extended representations $\widehat{\varrho} : \pi_1 X \longrightarrow \widehat{G} := G \rtimes \mathrm{Im}\,\varphi$.
- Set $H(\varrho) := \overline{\widehat{\varrho}(\pi_1 X)}^{\mathrm{Zar}} \subset \widehat{G}$.

Stability

Theorem (Cheng Shu, 2020)

The following conditions are equivalent:

1. $G \cdot \varrho$ is closed in $Z^1_\varphi(\pi_1 X; G)$.
2. $G \cdot \widehat{\varrho}$ is closed in $\mathrm{Hom}_\varphi(\pi_1 X; \widehat{G})$.
3. $H(\varrho)$ is a reductive subgroup of $\widehat{G}$.
4. $\widehat{\varrho}(\pi_1 X) \subset P$ parabolic in $\widehat{G} \Rightarrow \widehat{\varrho}(\pi_1 X) \subset L_P$ (Levi factor).

Untwisted case

- G a complex Lie group. A representation $\varrho : \pi_1 X \longrightarrow G$ gives rise to a flat G-bundle
$$\mathcal{E}_\varrho := (\widetilde{X} \times G)/\pi_1 X$$
- This induces an equivalence
$$\{\text{flat } G\text{-bundles}\} \longrightarrow \{G\text{-local systems}\}.$$
- Goal: extend to twisted G-local systems.

Bundles with connection

- G a complex Lie group. $\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(G)$ a group morphism.
- $\mathscr{G}_\varphi := [(\widetilde{X} \times G)/\pi_1 X]$ is now a holomorphic group bundle.
- Connection on a $\mathscr{G}_\varphi$-torsor $\mathcal{E}$ = splitting of
$$0 \longrightarrow \mathrm{ad}(\mathcal{E}) \longrightarrow \mathrm{At}(\mathcal{E}) \longrightarrow TY \longrightarrow 0$$
where $\mathrm{At}(\mathcal{E}) = \left\{\mathscr{G}_\varphi\text{–invariant vector fields on } \mathcal{E}\right\}$.

Integrable connections / De Rham space

- A connection on a $\mathscr{G}_\varphi$-torsor $\mathcal{E}$ is called *integrable*, or *flat*, if it is induced by a global section of the quotient sheaf $\underline{\mathcal{E}}/\mathscr{G}_\varphi$.

Theorem ("Riemann-Hilbert correspondence")

There is a bijection

$$\mathrm{H}^1_\varphi(\pi_1 X; G) \simeq \left\{\text{flat } \mathscr{G}_\varphi\text{–torsors } (\mathcal{E}, \nabla) \text{ on } Y\right\} \Big/ \text{isom.}$$

- Proof: uses the cover $p : X_\varphi \longrightarrow X$.
  Holonomy representations $\varrho : \pi_1 X_\varphi \longrightarrow G$ associated to $\mathrm{Im}\,\varphi$-invariant flat connections extend to *morphisms* $\widehat{\varrho} : \pi_1 X \longrightarrow G \rtimes \mathrm{Im}\,\varphi$.

Higgs bundles and local systems

- Goal: for G reductive, generalize to the twisted setting the Non-Abelian Hodge Correspondence (NAHC)

$$\{\text{local systems}\} \longleftrightarrow \{\text{semistable Higgs bundles}\}$$

  of Hitchin, Simpson, Donaldson and Corlette.
- Application: topology and geometry of twisted representation spaces.
- This also answers a question of Carlos Simpson's on the Dolbeault interpretation of the Betti space $\mathrm{H}^1_\varphi(\pi_1 X; G)$ in the case when $\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(G)$ is non-trivial (1992).

The Abelian case

Given a compact connected Riemann surface X of genus g, there is a homeomorphism

$$\underbrace{\mathrm{H}^1(X;\underline{\mathbb{C}^*})}_{\text{Betti space}} \simeq \underbrace{T^\vee \mathrm{Jac}(X)}_{\text{Dolbeault space}}$$

Intuition:

$$\begin{aligned} & \mathrm{H}^1(X;\underline{\mathbb{C}^*}) \\ \simeq\ & \mathrm{Hom}(\pi_1 X;\mathbb{C}^*) \\ \simeq\ & (\mathbb{C}^*)^{2g} \\ \simeq\ & (S_1 \times \mathbb{R})^{2g} \end{aligned}$$

$$\begin{aligned} & T^\vee \mathrm{Jac}(X) \\ \simeq\ & \mathrm{Jac}(X) \times \Omega^1_{\mathrm{hol}}(X;\mathbb{C}) \\ \simeq\ & (\mathbb{C}^g/\mathbb{Z}^{2g}) \times \mathbb{C}^g \\ \simeq\ & S_1^{2g} \times \mathbb{R}^{2g} \end{aligned}$$

Proof in the rank 1 case

$T^\vee \mathrm{Jac}(X) \simeq \mathrm{Jac}(X) \times \Omega^1_{\mathrm{hol}}(X;\mathbb{C})$ is a sub-space of

$$\mathrm{H}^1(X;\mathcal{O}_X^*) \times \mathrm{H}^0(X;\Omega^1_X).$$

Sketch of proof

The Abelian case of the NAHC (!) is obtained form the short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \stackrel{\exp}{\longrightarrow} \mathcal{O}_X^* \longrightarrow 1$$

and the Hodge decomposition theorem

$$\mathrm{H}^1(X;\underline{\mathbb{C}}) \simeq \mathrm{H}^1(X;\mathcal{O}_X) \times \mathrm{H}^0(X;\Omega^1_X).$$

Dolbeault moduli spaces

- Dolbeault space for $G = \mathbf{GL}(r;\mathbb{C})$:
  $$\mathrm{H}^1_{\mathrm{Dol}}\big(X;\mathbf{GL}(r;\mathbb{C})\big) := \{(\mathcal{E},\theta) \mid \theta \in \Omega^1_{\mathrm{hol}}\big(X;\mathrm{End}(\mathcal{E})\big)\}/\ \mathrm{isom}.$$
- $r = 1$ (line bundles) : $\mathrm{End}(\mathcal{L}) \simeq X \times \mathbb{C}$, so the Higgs field θ is just a holomorphic 1-form.
- For a reductive group G, a G-Higgs bundle is a pair (P, θ) where:
  - P is a holomorphic principal G-bundle,
  - $\theta \in \Omega^1_{\mathrm{hol}}(X; \mathrm{ad}(P))$.

Harmonic bundles

- Notion introduced by C. Simpson, serves as an intermediary between flat and Higgs bundles.
- Quadruple (E, h, A, ψ) where:
  - E is a $\mathcal{C}^\infty$ complex vector bundle.
  - h is a Hermitian metric on E.
  - A is a unitary connection and $\psi \in \Omega^1_{\mathcal{C}^\infty}\big(X; \mathrm{Herm}(E, h)\big)$.
- Hitchin equations:

$$\begin{aligned} F_A + \tfrac{1}{2}[\psi, \psi] &= 0 \\ d_A \psi &= 0 \\ d_A^* \psi &= 0 \end{aligned}$$

Non-Abelian Hodge Correspondence

$$\begin{array}{ccc} & (E, h, A, \psi) & \\ \swarrow & & \searrow \\ (E, \nabla := A + \psi) & \longleftarrow - - - \text{CHNA} - - - \longrightarrow & (E, \overline{\partial}_A, \theta := \psi^{1,0}) \end{array}$$

- Left-hand-side: $F_\nabla = 0$; (E, ∇) polystable.
- Right-hand-side: $\overline{\partial}_A \theta = 0$; $(E, \overline{\partial}_A, \theta)$ polystable.
- The main result is the existence of special Hermitian metrics (= harmonic metric) on these objects (Corlette, Simpson).
- Moduli spaces of stable objects are complex symplectic manifolds which are diffeomorphic but not biholomorphic.

Equivariant approach

- Let Y be a compact analytic orbi-curve of negative Euler characteristic. Assume given $\varphi : \pi_1 X \longrightarrow \mathrm{Aut}(G)$.
- Fix a presentation $Y \simeq [X/\Gamma]$ with Γ a finite group such that $\pi_1 X \subset \ker \varphi$. Denote by $p : X \longrightarrow Y$ the canonical projection.
- If $\mathcal{E}$ is a $\mathscr{G}_\varphi$-torsor on Y, then $p^*\mathcal{E}$ is a Γ-equivariant principal G-bundle on X:

$$\begin{array}{ccc} P & \xrightarrow{\tau_\gamma} & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{\gamma} & X \end{array}$$

with $\tau_{\gamma_1\gamma_2} = \tau_{\gamma_1}\tau_{\gamma_2}$ and $\tau_\gamma(p \cdot g) = \tau_\gamma(p) \cdot \varphi_\gamma(g)$.

Equivariant Higgs bundles

- Take (X, Γ) as above, with G reductive.
- A Γ-equivariant G-Higgs bundle is a triple (P, θ, τ) where:
  - P is a holomorphic principal G-bundle.
  - $\theta \in \Omega^1_{\mathrm{hol}}(X; \mathrm{ad}(P))$.
  - $\tau = (\tau_\gamma)_{\gamma \in \Gamma}$ is a Γ-equivariant structure on P, that leaves θ invariant.
- The orbi-bundle $\mathcal{E} := [P/\Gamma]$ on $Y := [X/\Gamma]$, endowed with the 1-form $\overline{\theta} \in \Omega^1_{\mathrm{hol}}(Y; \mathrm{ad}(\mathcal{E}))$ induced by θ, is a $\mathscr{G}_\varphi$-Higgs torsor (where $\mathscr{G}_\varphi = [(X \times G)/\Gamma]$).

Equivariant NAHC

- On a Riemann surface with symmetries (X, Γ), one can consider *Γ-invariant solutions of the Hitchin equations*.
- This gives rise to the following Γ-equivariant version of the NAHC, in which the Hermitian metric h is Γ-invariant:

$$(E, h, A, \psi, \Gamma)$$

$$(E, \nabla := A + \psi, \Gamma) \xleftarrow{\quad} \text{CHNA} \xrightarrow{\quad} (E, \overline{\partial}_A, \psi^{1,0}, \Gamma)$$

- The key point in the above is that the existence of a Γ-invariant harmonic metric is a property which is "invariant under taking finite covers".

ご清聴ありがとうございます！

Thank you for your attention! Merci !