Irregular conformal blocks and Painlevé equations

Hajime Nagoya

Kanazawa University

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Series representation of $\tau_{\mathrm{VI}}(t)$

In 2012, Gamayun, Iorgov and Lisovyy conjectured an expansion formula for the tau function of $\mathrm{P_{VI}}$ in terms of Virasoro conformal blocks with $c=1$:

$$\tau_{\mathrm{VI}}(t)=\sum_{n\in\mathbb{Z}}s^n C\left(\begin{matrix}\theta_1,\theta_t\\ \theta_\infty,\sigma+n,\theta_0\end{matrix}\right)\mathcal{F}\left(\begin{matrix}\theta_1,\theta_t\\ \theta_\infty,\sigma+n,\theta_0\end{matrix};t\right),$$

where $s,\sigma\in\mathbb{C}$, $\mathcal{F}(\theta,\sigma;t)=t^{\sigma^2-\theta_t^2-\theta_0^2}(1+O(t))$ is the 4-pt Virasoro conformal block with $c=1$, and

$$C(\theta,\sigma)=\frac{\prod_{\epsilon,\epsilon'=\pm}G(1+\theta_t+\epsilon\theta_0+\epsilon'\sigma)G(1+\theta_1+\epsilon\theta_\infty+\epsilon'\sigma)}{\prod_{\epsilon=\pm}G(1+2\epsilon\sigma)},$$

where $G(z)$ is the Barnes G-function such that $G(z+1)=\Gamma(z)G(z)$.

The AGT relation states

$$\mathcal{F}\left(\begin{matrix}\theta_1,\theta_t\\ \theta_\infty,\sigma,\theta_0\end{matrix};t\right)=t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{F}_{\lambda,\mu}\left(\begin{matrix}\theta_1,\theta_t\\ \theta_\infty,\sigma,\theta_0\end{matrix}\right)t^{|\lambda|+|\mu|},$$

where

$$\begin{aligned}&\mathcal{F}_{\lambda,\mu}\left(\begin{matrix}\theta_1,\theta_t\\ \theta_\infty,\sigma,\theta_0\end{matrix}\right)\\ &=\prod_{(i,j)\in\lambda}\frac{((\theta_t+\sigma+i-j)^2-\theta_0^2)((\theta_1+\sigma+i-j)^2-\theta_\infty^2)}{h_\lambda^2(i,j)(\lambda'_j+\mu_i-i-j+1+2\sigma)^2}\\ &\times(\lambda\to\mu,\sigma\to-\sigma).\end{aligned}$$

Here, $\lambda=(\lambda_1,\dots,\lambda_n)$ $(\lambda_i\geq\lambda_{i+1})$, $h_\lambda(i,j)$ is the hook length.

Jimbo's asymptotic formula

Jimbo gave the asymptotic expansion of the tau function (1982)

$$
\begin{aligned}
\tau(t) = \mathrm{const.}\ & t^{(\sigma^2-\theta_0^2-\theta_t^2)} \\
& \times \Bigg(1 + \frac{(\theta_0^2-\theta_t^2-\sigma^2)(\theta_\infty^2-\theta_1^2-\sigma^2)}{2\sigma^2} t \\
& \quad - \sum_{\epsilon=\pm} \frac{\hat{s}^\epsilon}{8\sigma^2(1+4\sigma^2)^2} (\theta_0^2-(\theta_t-\epsilon\sigma)^2)(\theta_\infty^2-(\theta_1-\epsilon\sigma)^2) t^{1+2\epsilon\sigma} \\
& \quad + \sum_{j=2}^{\infty} \sum_{|k|\le j} a_{jk} t^{j-2k\sigma} \Bigg),
\end{aligned}
$$

where $\hat{s}$, σ are expressed by the monodromy data of the linear equation associated with $\mathrm{P_{VI}}$.

The first part of $\tau(t)$:

$$t^{\sigma^2-\theta_0^2-\theta_t^2}\left(1+\frac{(\theta_0^2-\theta_t^2-\sigma^2)(\theta_\infty^2-\theta_1^2-\sigma^2)}{2\sigma^2}t\right)$$

is corresponding to the first part of the four point conformal block of Virasoro CFT with $c=1$

$$\begin{aligned}&\langle h_\infty|\Phi^{h_1}_{h_\infty,h_\sigma}(1)\Phi^{h_t}_{h_\sigma,h_0}(t)|h_0\rangle\\ =&t^{h_\sigma-h_t-h_0}\left(1+\frac{(h_\sigma+h_1-h_\infty)(h_\sigma+h_t-h_0)}{2h_\sigma}t+O(t^2)\right),\end{aligned}$$

where h_i are the conformal dimensions and $\Phi_{h_i}(z)$ are the primary fields with h_i [Belavin, Polyakov, Zamolodchikov, 1984].

Proofs

1. By constructing a fundamental solution to a Lax pair of P_{VI}, using the conformal field theory. [Iorgov, Lisovyy, Teschner, 2014]
2. By proving that the Fourier transform of conformal blocks satisfies the bilinear equations for P_{VI}, using embedding of direct sum of two Virasoro algebras to the sum of fermion and super Virasoro algebra. [Bershtein, Shchechkin, 2014]
3. Expansion of Fredholm determinant expression of the tau function for P_{VI}. [Gavrylenko, Lisovyy, 2016].

Monodromy preserving deformation

- $\mathrm{P_{VI}}$ is derived from monodromy preserving deformation of a Fuchsian system of rank 2 with four regular singular points.
- 4-point Virasoro conformal block with one degenerate field is the Gauss hypergeometric function.
- Connection problem of *m*-point Virasoro conformal block with one degenerate filed is reduced to connection problem of the Gauss hypergeometric function.
- A finite sub system of conformal blocks becomes a monodromy preserved fundamental solution of a Fuchsian system of rank 2 with four regular singular points. [Iorgov, Lisovyy, Teschner, 2014]

Ranks of singularities of corresponding 2 by 2 linear systems:

	singular points	ranks
$\mathrm{P_{VI}}$	$0, 1, t, \infty$	$(0, 0, 0, 0)$
$\mathrm{P_{V}}$	$0, t, \infty$	$(0, 0, 1)$
$\mathrm{P_{IV}}$	t, ∞	$(0, 2)$
$\mathrm{P_{III_1}}$	$0, t, \infty$	$(0, 0, 1/2)$
$\mathrm{P_{III_1}}$	$0, \infty$	$(1, 1)$
$\mathrm{P_{III_2}}$	$0, \infty$	$(1, 1/2)$
$\mathrm{P_{III_3}}$	$0, \infty$	$(1/2, 1/2)$
$\mathrm{P_{II}}$	t, ∞	$(0, 3/2)$
$\mathrm{P_{II}}$	∞	(3)
$\mathrm{P_{I}}$	∞	$(5/2)$

Ranks of singularities of conformal blocks:

	singular points	ranks	$t=0$	$t=\infty$
P_{VI}	$0,1,t,\infty$	$(0,0,0,0)$	[BPZ, 1984]	[BPZ, 1984]
P_V	$0,t,\infty$	$(0,0,1)$	[Gaiotto, 2009]	[N, 2015]
P_{IV}	t,∞	$(0,2)$		[N, 2015]
P_{III_1}	$0,t,\infty$	$(0,0,1/2)$	[Gaiotto, 2009]	[N, 2018]
P_{III_1}	$0,\infty$	$(1,1)$	[Gaiotto, 2009]	
P_{III_2}	$0,\infty$	$(1,1/2)$	[Gaiotto, 2009]	
P_{III_3}	$0,\infty$	$(1/2,1/2)$	[Gaiotto, 2009]	[GMS,2020]
P_{II}	t,∞	$(0,3/2)$		[N, 2018]
P_{II}	∞	(3)		[NU, 2019]
P_I	∞	$(5/2)$		

[BPZ]=[Belavin-Polyakov-Zamolodchikov],
[GMS]=[Gavrylenko-Marshakov-Stoyan],
[NU]=[Nishinaka-Uetoko].
Limiting procedures were proposed for conformal blocks of integer ranks [Gaiotto-Teschner 2012].

Irregular Verma Module

The Virasoro algebra $\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}c$ is the Lie algebra with

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{n^3-n}{12}c.$$

For $r \in \mathbb{Z}_{>0}$, define a module $M_\Lambda^{[r]}$ as a representation of Vir with a vector $|\Lambda\rangle$ such that

$$L_n|\Lambda\rangle = \Lambda_n|\Lambda\rangle \quad (n = r, r+1, \ldots, 2r),$$

with $\Lambda = (\Lambda_r, \ldots, \Lambda_{2r})$ and $M_\Lambda^{[r]}$ is spanned by linearly independent vectors of the form

$$L_{-\lambda} = L_{-i_1+r}\cdots L_{-i_k+r}|\Lambda\rangle \quad (i_1 \geq \cdots \geq i_k > 0).$$

We remark that $M_\Lambda^{[r]}$ is irreducible if and only if $\Lambda_{2r-1} \neq 0$ or $\Lambda_{2r} \neq 0$ [Lu, Guo and Zhao, 2011], [Felińska, Jaskólski and Kosztolowicz, 2012].

Vertex operator

Define a vertex operator

$$\Phi^{\Delta}_{\Lambda',\Lambda}(z) : M^{[r]}_{\Lambda} \to M^{[r]}_{\Lambda'}$$

by

$$[L_n, \Phi^{\Delta}_{\Lambda',\Lambda}(z)] = z^n \left(z\frac{d}{dz} + (n+1)\Delta \right) \Phi^{\Delta}_{\Lambda',\Lambda}(z),$$

$$\Phi^{\Delta}_{\Lambda',\Lambda}(z)|\Lambda\rangle = z^{\alpha} \exp\left(\sum_{n=1}^{r} \frac{\beta_n}{z^n} \right) \sum_{n=0}^{\infty} v_n z^n,$$

where $\alpha, \beta_n \in \mathbb{C}$, $v_n \in M^{[r]}_{\Lambda'}$ and $v_0 = |\Lambda'\rangle$.

Theorem (N, 2015)

If $\Lambda_{2r} \neq 0$, then the vertex operator $\Phi^{\Delta}_{\Lambda',\Lambda}(z)$: $M^{[r]}_{\Lambda} \to M^{[r]}_{\Lambda'}$ exists and is uniquely determined by the given parameters Λ, Δ, β_r.

Pairing

(I) $\mathrm{P_V}$ case.
A bilinear pairing $\langle\cdot\rangle$: $M^*_\Delta \times M^{[1]}_\Lambda \to \mathbb{C}$, ($\Lambda=(\Lambda_1,\Lambda_2)$) is uniquely defined by

$$\langle\Delta|\cdot|\Lambda\rangle = 1,$$
$$\langle u|L_n\cdot|v\rangle = \langle u|\cdot L_n|v\rangle \equiv \langle u|L_n|v\rangle,$$

where $\langle u| \in M^*_\Delta$, $|v\rangle \in M^{[1]}_\Lambda$. Because, L_n for $n \geq 1$ acts on $|\Lambda\rangle$ diagonally and L_n for $n \leq 0$ acts on $\langle\Delta|$ diagonally.
(II) $\mathrm{P_{IV}}$ case.
A bilinear pairing $\langle\cdot\rangle$: $V^*_0 \times M^{[2]}_\Lambda \to \mathbb{C}$, ($\Lambda=(\Lambda_2,\Lambda_3,\Lambda_4)$) is uniquely defined by

$$\langle 0|\cdot|\Lambda\rangle = 1,$$
$$\langle u|L_n\cdot|v\rangle = \langle u|\cdot L_n|v\rangle \equiv \langle u|L_n|v\rangle,$$

where $\langle u| \in V^*_0$, $|v\rangle \in M^{[2]}_\Lambda$. Here, V^*_0 is the irreducible highest weight module.

Theorem (Lisovyy-N-Roussillon, 2018)

A series expansion of the Painlevé V tau function at the irregular singular point ∞ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} s^n (-1)^{n(n+1)/2} G(1 \pm \theta_0 + \theta - \beta - n) G(1 + \theta_t \pm (\beta + n))$$
$$\times \langle \theta_0^2 | \cdot \left(\Phi^{\theta_t^2}_{(\theta-\beta-n,1/4),(\theta,1/4)}(t^{-1}) | (\theta, 1/4)\rangle \right).$$

Namely, $H = t(\log(t^{-2\theta_t^2-\theta^2/2} e^{-\theta t/2} \tau(t)))'$ satisfies the following differential equation (Hamiltonian DE for P_V)

$$(tH'')^2 - (H - tH' + 2(H')^2)^2 + \frac{1}{4}((2H' - \theta)^2 - 4\theta_0^2)((2H' + \theta)^2 - 4\theta_t^2) = 0.$$

We note that

$$\left(\langle \theta_0^2 | \Phi^{\theta_t^2}_{\theta_0^2, \sigma^2}(t^{-1}) \right) \cdot |(\theta, 1/4)\rangle$$

expresses a series expansion at the regular singular point 0.

Theorem (N)

A series expansion of the Painlevé IV tau function at the irregular singular point ∞ is given by

$$\tau(t) = t^{-2\theta_t^2} e^{\theta_t t^2/2} \sum_{n\in\mathbb{Z}} s^n G(1+\theta-\beta-n) G(1+\theta_t \pm (\beta+n)) \times \langle 0| \cdot \left(\Phi^{\theta_t^2}_{(\theta-\beta-n,0,1/4),(\theta,0,1/4)}(1/t) |(\theta,0,1/4)\rangle \right),$$

Namely, $H = (\log \tau(t))'$. Then, H satisfies the following differential equation (Hamiltonian DE for $\mathrm{P_{IV}}$)

$$(H'')^2 - (tH'-H)^2 + 4H'(H'-\theta-\theta_t)(H'-2\theta_t) = 0.$$

Here,

$$\langle 0| \cdot \left(\Phi^{\theta_t^2}_{(\theta-\beta,0,1/4),(\theta,0,1/4)}(1/t) |(\theta,0,1/4)\rangle \right) = t^{3\theta_t^2+\beta(2\theta-3\beta)} e^{\beta t^2/2} \sum_{n=0}^{\infty} a_m(\theta_t,\theta,\beta) t^{-2n}.$$

An irregular conformal block of type $(0,0,1)$ with $c=1$ is expanded as

$$\langle\theta_0^2|\cdot\left(\Phi^{\theta_t^2}_{(\theta,1/4),(\theta-\beta,1/4)}(t^{-1})|(\theta,1/4)\rangle\right)$$
$$=t^{2\theta_t^2+2\beta(\theta-\beta)}e^{\beta t}\left(1+2\left(2\beta^3-3\beta^2\theta+\beta\theta^2-\beta\theta_0^2-\beta\theta_t^2+\theta\theta_t^2\right)t^{-1}+\cdots\right)$$

It is natural to expect that this irregular conformal block has a combinatorial expression. For example, we find

$$2\left(2\beta^3-3\beta^2\theta+\beta\theta^2-\beta\theta_0^2-\beta\theta_t^2+\theta\theta_t^2\right)=2(\beta-\theta)\left(\beta^2-\theta_t^2\right)+2\beta\left((\theta-\beta)^2-\theta_0^2\right).$$

Thus, the coefficient of t^{-1} is successfully written as a sum of two factorized forms, which should be corresponding to pairs of partitions $((1),\emptyset)$, $(\emptyset,(1))$. Put

$$U_{\lambda,\emptyset}=\prod_{(i,j)\in\lambda}\frac{(2(\beta-\theta)+i-j)\left((\beta+i-j)^2-\theta_t^2\right)}{h_\lambda(i,j)^2}.$$

Then, the coefficient of t^{-2} is

$$\sum_{|\lambda|=2}\left(U_{\lambda,\emptyset}+U_{\emptyset,\lambda}\right)+2(2(\theta-\beta)\beta-1)\left(\beta^2-\theta_t^2\right)\left((\theta-\beta)^2-\theta_0^2\right).$$

The last term should be corresponding to $((1),(1))$.

Conjecture (N, arxiv:1611.08971)

An irregular conformal block of type $(0,0,1)$ *admits the following combinatorial formula*

$$\langle\theta_0^2|\cdot\left(\Phi^{\theta_t^2}_{(\theta,1/4),(\theta-\beta,1/4)}(t)|(\theta,1/4)\rangle\right)=\sum_{\substack{\lambda,\mu,\nu,\eta\in\mathbb{Y},\\ \nu\subset\lambda,\eta\subset\mu,|\nu|=|\eta|}} t^{|\lambda|+|\mu|}(-1)^{|\nu|}c^{\nu,\eta}_{\lambda,\mu}M_{\lambda/\nu,\mu/\eta}N_{\lambda,\mu},$$

where $c^{\nu,\eta}_{\lambda,\mu}\in\mathbb{Z}_{\geq 0}$,

$$M_{\lambda,\mu}=\prod_{(i,j)\in\lambda}(2(\beta-\theta)+i-j)\prod_{(i,j)\in\mu}(-2\beta+i-j),$$

$$N_{\lambda,\mu}=(-1)^{|\mu|}\prod_{(i,j)\in\lambda}\frac{(\beta+i-j)^2-\theta_t^2}{h_\lambda(i,j)^2}\prod_{(i,j)\in\mu}\frac{(\theta-\beta+i-j)^2-\theta_0^2}{h_\mu(i,j)^2}.$$

We observe

$$c^{\emptyset,\emptyset}_{\lambda,\mu}=1,\quad c^{(1),(1)}_{\lambda,\mu}=2|\lambda||\mu|,\quad c^{(2),(2)}_{\lambda,\mu}=q_\lambda q_\mu,\quad c^{(2),(1,1)}_{\lambda,\mu}=3q_\lambda q_{\mu'},\quad c^{\nu,\eta}_{\lambda,\mu}=c^{\eta,\nu}_{\mu,\lambda}=c^{\nu',\eta'}_{\lambda',\mu'},$$

$$q_\lambda=\lambda_1(\lambda_1-1)+\sum_{\substack{(i,j)\in\lambda,\\ i\neq 1}}\left(\lambda_1-1+\sum_{k=1}^{j-1}\lambda'_k\right),$$

and $c^{\nu,\eta}_{\lambda,\mu}\in\mathbb{Z}_{\geq 0}$ are determined uniquely if $|\lambda|+|\mu|\leq 5$.

Vertex operator of half rank

Define a vertex operator

$$\Phi^{\Delta}_{\Lambda',\Lambda}(z) : M^{[r]}_{\Lambda} \to M^{[r]}_{\Lambda'}$$

by

$$[L_n, \Phi^{\Delta}_{\Lambda',\Lambda}(z)] = z^n \left(z\frac{d}{dz} + (n+1)\Delta \right) \Phi^{\Delta}_{\Lambda',\Lambda}(z),$$

$$\Phi^{\Delta}_{\Lambda',\Lambda}(z)|\Lambda\rangle = z^{\alpha} \exp\left(\sum_{n=1}^{2r} \frac{\beta_n}{z^{n/2}} \right) \sum_{n=0}^{\infty} v_n z^{n/2},$$

where $\alpha, \beta_n \in \mathbb{C}$, $v_n \in M^{[r]}_{\Lambda'}$ and $v_0 = |\Lambda'\rangle$.

Proposition

If $\Lambda_{2r-1} \neq 0$ and $\Lambda_{2r} = 0$, then the vertex operator $\Phi^{\Delta}_{\Lambda',\Lambda}(z)$ exists, where $\Lambda' = \Lambda$, $\beta_{2r} = 0$, $v_m = \sum_{|\lambda|\leq m} c^{(m)}_{\lambda} L_{-\lambda}|\Lambda\rangle$ and $c^{(m)}_{\lambda}$ for any λ is a polynomial of α, $\beta_1, \dots, \beta_{2r-1}$, $\Lambda_r, \dots, \Lambda_{2r-1}$, Λ^{-1}_{2r-1} and $c^{(k)}_{\phi}$ $(k \leq m-1)$.

Since the vertex operator $\Phi^{\Delta}_{\Lambda',\Lambda}(z)$ depends on the parameters α, $\beta_1,\dots,\beta_{2r-1}$, $\Lambda_r,\dots,\Lambda_{2r-1}$, Λ^{-1}_{2r-1} and complex numbers $c^{(k)}_{\phi}$, it should be denoted by $\Phi^{\Delta}_{\Lambda,\Lambda}(\alpha,\beta,c_\phi;z)$.
We define actions of L_{-n} for any positive integer n on a linear operator $\Phi(z)$ as follows.

$$
\begin{aligned}
L_{-1}\cdot\Phi(z) =& \frac{\partial}{\partial z}\Phi(z)\\
L_{-n}\cdot\Phi(z) =& :\frac{1}{(n-2)!}\partial_z^{n-2}(T(z))\Phi(z):\\
=& \frac{1}{(n-2)!}\left(\partial_z^{n-2}(T_-(z))\Phi(z)+\Phi(z)\partial_z^{n-2}(T_+(z))\right),
\end{aligned}
$$

where $n\geq 2$ and

$$
T_-(z)=\sum_{n\leq -2}L_n z^{-n-2},\quad T_+(z)=\sum_{n\geq -1}L_n z^{-n-2}.
$$

We define descendants of the vertex operator $\Phi_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; v, z)$ for $v \in M_\Delta$ by

$$\Phi_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; |\Delta\rangle, z) = \Phi^{\Delta}_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; z),$$
$$\Phi_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; L_{-\lambda}|\Delta\rangle, z) = L_{-\lambda} \cdot \Phi^{\Delta}_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; z).$$

A singular vector χ is a vector such that

$$L_n \chi = 0, \quad (n \geq 1).$$

It is known [Kac, 1979], [Feigin, Fuchs, 1982] that for a positive integer p, q, a singular vector $\chi_{p,q}$ of level pq exists in $M_{\Delta_{p,q}}$, where

$$c = 13 - 6\left(t + \frac{1}{t}\right), \quad \Delta_{p,q} = \frac{(pt-q)^2 - (t-1)^2}{4t}.$$

Definition

For $p, q \in \mathbb{Z}_{>0}$, an irregular vertex operator $\Phi^{\Delta_{p,q}}_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; z)$ is called singular if it satisfies

$$\Phi^{\Delta_{p,q}}_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; \chi_{p,q}, z) = 0$$

for a singular vector $\chi_{p,q}$ of level pq in $M_{\Delta_{p,q}}$.

By definition the singular vertex operator $\Phi^{\Delta_{p,q}}_{\Lambda,\Lambda}(\alpha, \beta, c_\phi; z)$ satisfies a kind of linear differential equation.

Conjecture (N, arXiv:1804.04782)

For any positive integers p, q, there exist singular vertex operators $\Phi_{\Lambda,\Lambda}^{\Delta_{p,q}}(\alpha, \beta, c_\phi; z)$ such that α, β_1, ..., β_{2r-1} and $c_\phi^{(m)}$ $(m \in \mathbb{Z}_{\geq 1})$ are solved as polynomials of c, Λ_r, ..., Λ_{2r-1}, Λ_{2r-1}^{-1}. Moreover, the number of such sets $\{\alpha, \beta, c_\phi\}$ with multiplicity is pq.

We may denote a singular vertex operator by $\Phi_{\Lambda,\Lambda}^{\Delta_{p,q},i}(z)$ $(i = 1, \ldots, pq)$.

Conjecture (N, arXiv:1804.04782)

There exist a unique vertex operator $\Phi_{\Lambda,\Lambda}^{\Delta_{p,q}}(\alpha, \beta, c_\phi; z)$ such that the coefficients $c_\lambda^{(m)}$ of it are polynomials of c, Δ, β_{2r-1}, Λ_r, ..., Λ_{2r-1}, Λ_{2r-1}^{-1} and it is equal to the singular vertex operator $\Phi_{\Lambda,\Lambda}^{\Delta_{p,q},i}(z)$ when $\beta_{2r-1} = \beta_{2r-1}^{p,q,i}$ and $\Delta = \Delta_{p,q}$.

We denote such irregular vertex operator as $\Phi_{\Lambda,\Lambda}^{\Delta,\beta_{2r-1}}(z)$ and call it irregular vertex operator of rank $r/2$.

The first few terms of irregular vertex operator of rank $1/2$ with $\Lambda_1 = 1$, $\beta_1 = \beta$:

$$\alpha = \frac{\beta^2}{32} - \frac{3\Delta}{2}, \quad v_1 = \frac{\beta^3}{256} + \frac{\beta}{64}(c - 4\Delta + 1) - \frac{\beta}{2} L_0,$$

$$\begin{aligned} v_2 = {} & \frac{\beta^6}{131072} + \frac{\beta^4(c - 4\Delta + 6)}{16384} \\ & + \frac{\beta^2\left(3c^2 - 24c\Delta + 74c + 48\Delta^2 - 168\Delta + 103\right)}{24576} + \frac{\Delta}{64}(\Delta - c - 2) \\ & + \left(-\frac{\beta^4}{512} - \frac{\beta^2}{128}(c - 4\Delta + 13) + \frac{\Delta}{2}\right) L_0 + \frac{\beta^2}{8} L_0^2. \end{aligned}$$

We note that the central charge c appears in v_1, which do not happen in integer rank case. Gavrylenko, Marshakov, Stoyan also reported the same fact for their irregular conformal block of type $(1/2, 1/2)$.

Conjecture (N, arXiv:1804.04782)

A series expansion of the Painlevé III_1 *tau function at the irregular singular point* ∞ *is given by*

$$\tau(t) = t^{-\theta_1\theta_2} e^{-t/2} \sum_{n\in\mathbb{Z}} s^n 2^{-(\nu+n)^2} G(1+\nu+n\pm(\theta_1+\theta_2)/2)$$
$$\times \langle(\theta_1-\theta_2)^2/4| \cdot \left(\Phi^{(\theta_1+\theta_2)^2/4,4(\nu+n)}_{(1,0),(1,0)}(t^{-1})|(1,0)\rangle\right).$$

Namely, $H = t(\log(\tau(t)))'$ *satisfies the following differential equation (Hamiltonian DE for* $\mathrm{P}_{\mathrm{III}_1}$*)*

$$(tH'')^2 - (4(H')^2-1)(H-tH') + 4\theta_1\theta_2 H' - (\theta_1+\theta_2)^2 = 0.$$

We have a similar conjecture for the Painlevé II tau function.

Other results

- A series expansion of the tau function of q-P_{VI} [Jimbo-N-Sakai, 2017]
- Series expansions of the tau functions of q-P_{V} and q-$\mathrm{P}_{\mathrm{III}}$ [Matsuhira-N, 2019]
- A series expansion of the tau function of q-FST system [N, 2021]
- Irregular vertex operators of a super Virasoro algebra (NSR algebra)

In 2014, Bershtein, Shchechkin showed that using embedding of direct sum of two Virasoro algebras to the sum of fermion and a super Virasoro algebra, the Fourier transform of 4-pt regular conformal blocks discovered by Gamayun, Iorgov, Lisovyy satisfies the bilinear equations for P_{VI}.

The Neveu-Schwarz-Ramond algebra

$$\mathrm{NSR} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{r\in\mathbb{Z}+\frac{1}{2}} G_r \oplus \mathbb{C}c$$

is an algebra with (anti) commuting relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$
$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r},$$
$$[G_r, G_s]_+ = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.$$

For $\Lambda = (\Lambda_p, \dots, \Lambda_{2p}) \in \mathbb{C}^p \times \mathbb{C}^*$, let $M_\Lambda^{[p]}$ be an irregular Verma module of NSR such that

$$L_n|\Lambda\rangle = \Lambda_n|\Lambda\rangle \quad (n \geq p), \quad G_r|\Lambda\rangle = 0 \quad \left(r > p - \frac{1}{2}\right),$$

where $\Lambda_n = 0$ if $n > 2p$.

Definition

We define irregular vertex operators $\Phi^{\Delta}_{\Lambda',\Lambda}(z)$ *and* $\Psi^{\Delta}_{\Lambda',\Lambda}(z) : M^{[p]}_{\Lambda} \to M^{[p]}_{\Lambda'}$ *by*

$$\left[L_n, \Phi^{\Delta}_{\Lambda',\Lambda}(z)\right] = z^n \left(z\frac{\partial}{\partial z} + (n+1)\Delta \right) \Phi^{\Delta}_{\Lambda',\Lambda}(z),$$

$$\left[L_n, \Psi^{\Delta}_{\Lambda',\Lambda}(z)\right] = z^n \left(z\frac{\partial}{\partial z} + (n+1)\left(\Delta + \frac{1}{2}\right) \right) \Psi^{\Delta}_{\Lambda',\Lambda}(z),$$

$$\left[G_r, \Phi^{\Delta}_{\Lambda',\Lambda}(z)\right] = z^{r+\frac{1}{2}} \Psi^{\Delta}_{\Lambda',\Lambda}(z),$$

$$\left[G_r, \Psi^{\Delta}_{\Lambda',\Lambda}(z)\right]_+ = z^{r-\frac{1}{2}} \left(z\frac{\partial}{\partial z} + (2r+1)\Delta \right) \Phi^{\Delta}_{\Lambda',\Lambda}(z),$$

$$\Phi^{\Delta}_{\Lambda',\Lambda}(z)|\Lambda\rangle = z^{\alpha} \exp\left(\sum_{i=1}^{p} \frac{\beta_i}{z^i} \right) \sum_{m=0}^{\infty} v_m z^{\frac{m}{2}},$$

$$\Psi^{\Delta}_{\Lambda',\Lambda}(z)|\Lambda\rangle = z^{\alpha-\frac{p+1}{2}} \exp\left(\sum_{i=1}^{p} \frac{\beta_i}{z^i} \right) \sum_{m=0}^{\infty} u_m z^{\frac{m}{2}},$$

where $v_0 = u_0 = |\Lambda'\rangle$, $v_m, u_m \in M^{[p]}_{\Lambda'}$.

Theorem

If $\Lambda_{2p} \neq 0$, *then the irregular vertex operators* $\Phi^{\Delta}_{\Lambda',\Lambda}(z)$ *and* $\Psi^{\Delta}_{\Lambda',\Lambda}(z) : M^{[p]}_{\Lambda} \to M^{[p]}_{\Lambda'}$ *exists and are uniquely determined by the given parameters* Λ, Δ, β_p.

The $\mathrm{F} \oplus \mathrm{NSR}$ algebra

$$\mathrm{F} \oplus \mathrm{NSR} = \bigoplus_{n\in\mathbb{Z}+\frac{1}{2}} \mathbb{C} f_r \oplus \bigoplus_{n\in\mathbb{Z}} \mathbb{C} L_n \oplus \bigoplus_{r\in\mathbb{Z}+\frac{1}{2}} G_r \oplus \mathbb{C} c$$

is an algebra with (anti) commuting relations

$$\begin{aligned}
&[f_r, f_s]_+ = \delta_{r+s,0}, \quad [f_r, G_S]_+ = [f_r, L_n] = 0, \\
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
&[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \\
&[G_r, G_s]_+ = 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.
\end{aligned}$$

Put

$$c = 1 + 2Q^2 \quad (Q = b^{-1} + b).$$

For $\Lambda^{\mathrm{NS}} = (\Lambda_p^{\mathrm{NS}}, \dots, \Lambda_{2p}^{\mathrm{NS}}) \in \mathbb{C}^p \times \mathbb{C}^*$, let $\pi_{\mathrm{F}\oplus\mathrm{NSR}}^{\Lambda,[p]}$ be a tensor product of a Verma module π_{F} and an irregular Verma module $\pi_{\mathrm{NSR}}^{\Lambda^{\mathrm{NS}},[p]}$ such that

$$f_r|1\rangle = 0 \quad (r > 0), \quad L_n|\Lambda^{\mathrm{NS}}\rangle = \Lambda_n^{\mathrm{NS}}|\Lambda^{\mathrm{NS}}\rangle, \quad G_r|\Lambda^{\mathrm{NS}}\rangle = 0 \quad (n, r \geq p),$$

where $\Lambda_n^{\mathrm{NS}} = 0$ if $n > 2p$. The irregular vector of $\pi_{\mathrm{F}\oplus\mathrm{NSR}}^{\Lambda,[p]}$ is $|1\rangle \otimes |\Lambda^{\mathrm{NS}}\rangle$.
We use the free-field realization of the NSR algebra generated by c_n $(n \in \mathbb{Z})$ and ψ_r $(r \in \mathbb{Z} + \frac{1}{2})$ with relations

$$[c_n, c_m] = n\delta_{n+m,0}, \quad [c_n, \psi_r] = 0, \quad [\psi_r, \psi_s]_+ = \delta_{r+s,0}.$$

For $P = (P_0, \dots, P_p) \in \mathbb{C}^{p+1}$, let $\mathcal{F}_P^{[p]}$ be an irregular Fock module of this algebra generated by a vacuum vector $|P\rangle$ such that

$$c_n|P\rangle = P_n|P\rangle \quad (n = 0, \dots, p), \quad c_n|P\rangle = \psi_r|P\rangle = 0 \quad (n > p, \ r > 0).$$

We have two free field representations of the same irregular Verma module of NSR:

$$L_n = \frac{1}{2}\sum_{k\neq 0,n} c_k c_{n-k} + \frac{1}{2}\sum_r \left(r - \frac{n}{2}\right)\psi_{n-r}\psi_r + \frac{i}{2}\left(Qn - 2c_0\right)c_n \quad (n \neq 0),$$

$$L_0 = \sum_{k>0} c_{-k}c_k + \sum_{r>0} r\psi_{-r}\psi_r + \frac{1}{2}\left(\frac{Q^2}{4} - c_0^2\right),$$

$$G_r = \sum_{k\neq 0} c_k\psi_{r-k} + i(Qr - c_0)\psi_r,$$

and

$$L_n = \frac{1}{2}\sum_{k\neq 0,n} c_k c_{n-k} + \frac{1}{2}\sum_r \left(r - \frac{n}{2}\right)\psi_{n-r}\psi_r - \frac{i}{2}\left(Q(n-2p) + 2c_0\right)c_n \quad (n \neq 0),$$

$$L_0 = \sum_{k>0} c_{-k}c_k + \sum_{r>0} r\psi_{-r}\psi_r + \frac{1}{2}\left(\frac{Q^2}{4} - (Qp - c_0)^2\right),$$

$$G_r = -\sum_{k\neq 0} c_k\psi_{r-k} + i(Q(r-p) + c_0)\psi_r.$$

In both free field representations, the actions of L_n $(n \geq p)$ and G_r $(r > p)$ on $|P\rangle$ are

$$L_n|P\rangle = \frac{1}{2}\sum_{k=n-p}^{p} P_k P_{n-k}|P\rangle \quad (p < n \leq 2p),$$

$$L_p|P\rangle = \frac{1}{2}\sum_{k=1}^{p-1} P_k P_{p-k}|P\rangle + \frac{i}{2}(Qp - 2P_0)P_p|P\rangle,$$

$$L_n|P\rangle = 0 \quad (n > 2p), \quad G_r|P\rangle = 0 \quad (r > p).$$

Hence,

$$\Lambda_n^{\mathrm{NS}} = \frac{1}{2}\sum_{k=n-p}^{p} P_k P_{n-k} \quad (p < n \leq 2p),$$

$$\Lambda_p^{\mathrm{NS}} = \frac{1}{2}\sum_{k=1}^{p-1} P_k P_{p-k} + \frac{i}{2}(Qp - 2P_0)P_p.$$

The embedding of the $\mathrm{Vir} \oplus \mathrm{Vir}$ subalgebra in the $\mathrm{F} \oplus \mathrm{NSR}$ algebra is defined by

$$L_n^{(1)} = \frac{b^{-1}}{b^{-1} - b} L_n - \frac{b^{-1} + 2b}{2(b^{-1} - b)} \sum_{\mathbb{Z}-\frac{1}{2}} r : f_{n-r} f_r : + \frac{1}{b^{-1} - b} \sum_{\mathbb{Z}-\frac{1}{2}} f_{n-r} G_r,$$
$$L_n^{(2)} = \frac{b}{b - b^{-1}} L_n - \frac{b + 2b^{-1}}{2(b - b^{-1})} \sum_{\mathbb{Z}-\frac{1}{2}} r : f_{n-r} f_r : + \frac{1}{b - b^{-1}} \sum_{\mathbb{Z}-\frac{1}{2}} f_{n-r} G_r.$$

$L_n^{(\eta)}$ $(\eta = 1, 2)$ act on $|P\rangle$ as follows.

$$L_n^{(1)} |P\rangle = \frac{b^{-1}}{b^{-1} - b} \Lambda_n^{\mathrm{NS}} |P\rangle \quad (p \leq n \leq 2p),$$
$$L_n^{(2)} |P\rangle = \frac{b}{b - b^{-1}} \Lambda_n^{\mathrm{NS}} |P\rangle \quad (p \leq n \leq 2p).$$

Put

$$\chi_r^{(j)} = f_r - i\psi_r^{(j)} \quad (j = 1, 2).$$

We define the vectors $|P, n\rangle$ $(2n \in \mathbb{Z})$ as follows.

$$|P, 0\rangle = |1\rangle \otimes |P\rangle = |1\rangle \otimes |\Lambda^{\mathrm{NS}}\rangle, \quad |P, n\rangle = \prod_{r=1/2}^{(4n-1)/2} \chi_{-r}^{(1)}|1\rangle \otimes |\Lambda^{\mathrm{NS}}\rangle \quad (n > 0),$$

$$|P, n\rangle = \prod_{r=1/2}^{(-4n-1)/2} \chi_{-r}^{(2)}|1\rangle \otimes |\Lambda^{\mathrm{NS}}\rangle \quad (n < 0).$$

Straightforward calculations show that

$$L_n^{(1)}|P, m\rangle = \left(\frac{b^{-1}}{b^{-1} - b} \Lambda_n^{\mathrm{NS}} - \delta_{n,p} \frac{2im}{b^{-1} - b} P_p \right) |P, m\rangle,$$

$$L_n^{(2)}|P, m\rangle = \left(\frac{b}{b - b^{-1}} \Lambda_n^{\mathrm{NS}} - \delta_{n,p} \frac{2im}{b - b^{-1}} P_p \right) |P, m\rangle$$

for $m > 0$.

We can write $|P, m\rangle = |\Lambda^{(m,1)}\rangle \otimes |\Lambda^{(m,2)}\rangle$.

Theorem (N)

For $P_p \neq 0$, $\pi^{\Lambda^{\mathrm{NS}},[p]}_{\mathrm{F}\oplus\mathrm{NSR}}$ is isomorphic to the sum of irregular Verma modules over $\mathrm{Vir} \oplus \mathrm{Vir}$

$$\pi^{\Lambda^{\mathrm{NS}},[p]}_{\mathrm{F}\oplus\mathrm{NSR}} \cong \bigoplus_{2m\in\mathbb{Z}} \pi^{m,[p]}_{\mathrm{Vir}\oplus\mathrm{Vir}}.$$

The irregular vectors of the irregular Verma module $\pi^{m,[p]}_{\mathrm{Vir}\oplus\mathrm{Vir}}$ of $\mathrm{Vir} \oplus \mathrm{Vir}$ are $|\Lambda^{(m,1)}\rangle \otimes |\Lambda^{(m,2)}\rangle$.

This result should lead to a decomposition of an irregular vertex operator of NSR algebra into the sum of irregular vertex operators of $\mathrm{Vir} \oplus \mathrm{Vir}$. We expect that for $p = 1, 2$, an expectation value of an irregular vertex operator of NSR algebra satisfies the bilinear equation for $\mathrm{P_V}$, $\mathrm{P_{IV}}$, respectively.