# Some Aspects and applications of non-generic isomonodromy deformations

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**Isomonodromy Deformations**

Riemann (1857). Isomonodromy deformations: find functions with varying regular singularities and constant monodromy. ($\longrightarrow$ birth of Riemnann-Hilbert problems).

R. Fuchs (1907), Schlesinger, Garnier (1912). Study of ODEs with prefixed monodromy, Fuchsian syetems, Schlesinger equations, Painlevé equations.

Jimbo, Miwa, Ueno (1981). Systematic treatment of *generic isomonodromy deformations* (generic = differential systems have generic matrices, e.g. non-resonant, diagonalizable, etc)

From the analytic viewpoint....

**Approach 1.** [Jimbo-Miwa-Ueno] Given a differential system

$$\frac{dY}{dz} = A(z,u)Y, \qquad A(z,u) \text{ is rational of } z \in \mathbb{C}, \text{ analytic of } u \in \mathbb{C} \text{ polydisc}, \qquad (1)$$

prove that a class of fundamental matrix solutions with "canonical form" have constant "essential" monodromy data (monodromy matrices, monodromy exponents) if and only if they satisfy a Pfaffian system

$$dY = \omega(z,u)Y, \qquad \omega \text{ is a } 1\text{−form in } dz,\ du_1,\ ...,\ du_n. \qquad (2)$$

with $\omega(z,u)$ determined by $A(z,u)$.

**Approach 2.** A Pfaffian system (2) is given, satisfying the Frobenius integrability condition

$$d\omega = \omega \wedge \omega, \qquad \omega(z,u)\Big|_{u \text{ fixed}} = A(z,u)dz.$$

$\Longrightarrow \exists$ fundamental matrix solution $Y(z,u)$ of (2) with constant monodromy w.r.t. $z$.

Depending on $\omega(z,u)$, prove if the Pfaffian system admits fundamental matrix solutions with a canonical structure and constant "essential" monodromy data.

Today's seminar...

We consider a specific case: a linear $n \times n$ differential system:

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, \qquad \Lambda(u) = \mathrm{diag}(u_1, \dots, u_n),$$

*A*($u$) *holomorphic* of $u = (u_1, \dots, u_n)$ in a polydisc $\mathbb{D}$.

Some eigenvalues may coalesce in $\mathbb{D}$

$$u_j - u_k \to 0 \qquad \text{for some } j \neq k.$$

• The purpose of the talk is to describe how the isomonodromy deformation theory can be extended to include the case of coalescing eigenvalues.

This is a "non-admissible deformation" in the standard isomonodromy deformation theory of Jimbo-Miwa-Ueno, Fokas-Its-Kapaev-Novokshenov.

Is it possible to compute fundamental matrix solutions and *monodromy data* valid for the whole $\mathbb{D}$ ?

• Some applications: Frobenius manifolds, Painlevé equations.

System

$$\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, \qquad \Lambda(u) = \mathrm{diag}(u_1, \ldots, u_n),$$

is important in several respects.

- It is related to Fuchsian systems by Laplace transform [Balser-Jurkat-Lutz, Schäfke]

$$Y(z) = \int_\gamma e^{\lambda z} \Psi(\lambda) d\lambda, \qquad \gamma \text{ is a suitable path,}$$

$$\frac{d\Psi}{d\lambda} = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} \Psi, \qquad B_k := -E_k(A + I)$$

Coalescence of eigenvalues $\longleftrightarrow$ coalescence of Fuchsian singularities.

- $\underline{n = 3}$.

In some cases, isomonodromy deformation equations for $A(u)$ are equivalent to the Painlevé VI equation

$$\frac{d^2 y(x)}{dt^2} = F(x, y, \frac{dy}{dx})$$

$F$ = quadratic polyn. in $\frac{dy}{dx}$ and rational in $(x, y)$.

$F$ has poles in $x = 0, 1, \infty$.

The entries of $A(u)$ are *explicit* algebraic functions of a Painlevé transcencents $y(x)$

[Harnad 1994; Dubrovin 1996; Mazzocco 2002; Boalch 2004; Degano & DG 2021]

Singularities $x = 0, 1, \infty$ correspond to coalescing eigenvalues through

$$x = \frac{u_2 - u_1}{u_3 - u_1}.$$

- **Analytic theory of semisimple Dubrovin-Frobenius Manifolds** [Dubrovin, Hertling, Manin, etc]

  An analytic manifold $M$ of dimension $n$, with a flat metric $\eta$.
  Any tangent space $T_pM$ is a Frobenius algebra (product $\cdot$ , commutative, associative with unit) such that

  $$\eta(X\cdot Y, Z)=\eta(X, Y\cdot Z), \quad X, Y, Z \text{ tangent vectors}$$

  In *flat coordinates* $t=(t^1,\dots,t^n)$

  $$\eta\left(\frac{\partial}{\partial t^\alpha}\cdot\frac{\partial}{\partial t^\beta},\frac{\partial}{\partial t^\gamma}\right)=\frac{\partial^3 F}{\partial t^\alpha\partial t^\beta\partial t^\gamma}, \quad \alpha,\beta,\gamma=1,\dots,n.$$

  satisfying the *WDVV* equations of associativity of $2D$-topological field theory:

  $$\partial_\alpha\partial_\beta\partial_\gamma F\ \eta^{\gamma\rho}\ \partial_\rho\partial_\mu\partial_\nu F\ =\ \text{the same with exchange } \alpha\leftrightarrow\nu.$$

  $$\eta^{\gamma\rho} \text{ are matrix entries of } \left(\eta(\partial_\alpha,\partial_\beta)\right)^{-1}, \quad \partial_\alpha:=\frac{\partial}{\partial t^\alpha}.$$

  *Note:* Precise and complete definitions are omitted here...

**Examples**

• Singularity theory: $M$ is space of versal deformation of simple singularities.

*Example*: $M = \{f(x;a) = x^{n+1} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\}$

• Orbit spaces $M = \mathbb{C}^n/W$, $W$ = Coxeter group. Analogous cases with extende affine Coxeter groups, Jacobi groups, etc.

• Quantum Cohomology $QH^\bullet(X)$ as a $t$-deformation of the classical cohomology $H^\bullet(X,\mathbb{C})$ of a smooth projective variety $X$, related to Gromov-Witten invariants theory
(the cup-product is deformed, so that the cohomology algebra becomes semisimple; $F(t)$ generating function of Gromov-Witten invariants of genus zero).

**Important characterisation** [Dubrovin]. *A manifold $M$ is Frobenius if and only if there is on $M$ a certain family of flat connections depending on parameter $z \in \mathbb{C}$.*

If $(T_pM, \cdot)$ is semisimple (no nilpotents) on open-dense subset of $M$, there are local canonical coordinates $(u_1, \ldots, u_n)$, $u_j \neq u_k$ for all $j \neq k$.

Fatness is equivalent to the isomonodromy deformation equations of

$$\frac{\partial Y}{\partial z} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, \quad A \text{ skew-symmetric.}$$

Locally constant monodromy data parameterise local charts of the Frobenius manifold. From one chart to another through explicit action of braid group on data.

Given the differential system

→ Compute monodromy data of a chart

→ Arithmetically compute data for another chart by braid group action

→ From the new data, reconstruct the system (i.e. $A(u)$) in another chart through a Riemann-Hilbert inverse problem.

• This gives analytic continuation of Frobenius structure, because from a specific fundamental $n \times n$ matrix solution (*Levelt form*)

$$Y(z,u) = \Big(\sum_{p=0}^{\infty} \phi_p(u) z^p\Big) z^D z^L,$$

there are explicit parametric formulae allowing to reconstruct the local Frobeniius structure

$$t_\alpha(u) := \eta_{\alpha\beta} t^\beta = \sum_{i=1}^{N} \phi_{i\alpha,0}(u)\phi_{i1,1}(u),$$

$$F(t(u)) = \frac{1}{2}\left\{ t^\alpha t^\beta \sum_{i=1}^{n} \phi_{i\alpha,0}(u)\ \ \phi_{i\beta,1}(u) - \sum_{i=1}^{N} \left(\phi_{i1,1}(u)\phi_{i1,2}(u) + \phi_{i1,3}(u)\phi_{i1,0}(u)\right) \right\}$$

**Back to our case:** Isomonodromic

$$(DS) \qquad \frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y, \qquad A(u) \text{ holomorphic in polydisc } \mathbb{D}$$

1. $\mathbb{D}(u^0) := \{u \in \mathbb{C}^n \mid \max_j |u_j - u_j^0| < \epsilon\}$, $\quad u_j^0 \neq u_k^0 \ \forall j \neq k$.
   **Admissible deformations** [Jimbo-Miwa-Ueno]: just compute the monodromy data of
   $$\frac{dY}{dz} = \left(\Lambda(u^0) + \frac{A(u^0)}{z}\right) Y.$$

2. $\mathbb{D}(u^c)$, $\quad u_j^c = u_k^c$ for some $j \neq k$.
   **Non-admissible deformations**. We have *coalescence of eigenvalues of* $\Lambda$ at a
   $$\text{coalescence locus} \quad \Delta := \mathbb{D}(u^c) \cap \left(\bigcup_{j\neq k} \{u_j - u_k = 0\}\right).$$
   Problems with fundamental matrix solutions may occur at $\Delta$. It is not clear if and when we can define and compute monodromy data for the whole $\mathbb{D}(u^c)$ starting from
   $$\frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y.$$

*Example of problematic issues.*

$$\frac{dY}{dz}=\left[\begin{pmatrix}0&0\\0&u\end{pmatrix}+\frac{1}{z}\begin{pmatrix}1&0\\u&2\end{pmatrix}\right]Y.$$

The coalescence locus is $\Delta=\{u=0\}$.

$$Y(z,u)=\widehat{Y}(z,u)\begin{pmatrix}z&0\\0&z^2e^{uz}\end{pmatrix}\qquad \text{canonical behaviour}$$

$$\widehat{Y}(z,u)\sim I+\sum_{k=2}^{\infty}\frac{(-1)^k k!}{u^{k-1}}\,\frac{1}{z^k},\quad z\to\infty,\quad -\frac{3\pi}{2}<\arg(zu)<\frac{3\pi}{2}$$

- Coefficients of asymptotic expansion have poles at $\Delta$.
- $\Delta$ is a branching locus:

$$\widehat{Y}(z,u)=I+\begin{pmatrix}0&0\\u\Big[uz\,e^{uz}\mathrm{Ei}(uz)-1\Big]&0\end{pmatrix},\quad \mathrm{Ei}(\xi):=\int_{\xi}^{+\infty}x^{-1}e^{-x}dx$$

$$\mathrm{Ei}(uz)=-\log(zu)-\gamma+\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{m\cdot m!}(uz)^m,\quad uz\to 0.$$

- $\lim_{u\to 0}Y(z,u)$ is not defined. So, it is not equal to solution of system with $u=0$:

$$Y_a(z)=\left[I+\frac{1}{z}\begin{pmatrix}1&a\\0&1\end{pmatrix}\right]\begin{pmatrix}z&0\\0&z^2\end{pmatrix},\qquad a\in\mathbb{C}.$$

## The generic case of Jimbo, Miwa, Ueno

Deformations in $\mathbb{D}(u^0) = \left\{ u \in \mathbb{C}^n \;\middle|\; \max_{1\le j\le n} |u_j - u_j^0| \le \epsilon_0 \right\}$ polydisc at $u^0$.

No coalescence points

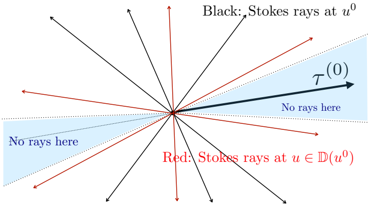
Solutions at $z = \infty$

- **Stokes rays of $\Lambda(u^0)$**

$$\Re((u_j^0 - u_k^0)z) = 0, \quad \Im((u_j^0 - u_k^0)z) < 0.$$

- **Admissible direction at $u^0$: $\arg z = \tau^{(0)}$**

not coinciding with any of the Stokes rays above.




- **Stokes rays of $\Lambda(u)$**

$$\Re((u_j - u_k)z) = 0, \quad \Im((u_j - u_k)z) < 0.$$

Stokes rays of $\Lambda(u)$ don't cross admissible direction $(\text{mod } \pi)$, as $u$ varies in $\mathbb{D}(u^0)$ small.

A classical result (Sibuya, Wasow, etc). $\mathbb{D}(u^0)$ small.

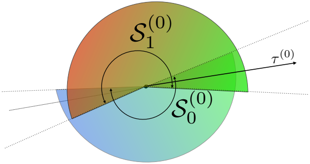
**1)** $\exists$ unique formal fundamental matrix solution

$$Y_F(z,u) = \Big(I + \sum_{\ell=1}^{\infty} F_\ell(u) z^{-\ell}\Big) z^{B(u)} e^{z\Lambda(u)}, \quad B(u) := \text{diag}(A_{11}, \dots, A_{nn});$$

$F_\ell(u)$ holomorphic in $\mathbb{D}(u^0)$ and recursively computable.

**2)** $\forall\ \nu \in \mathbb{Z}$, $\exists\ \delta > 0$ small

and sectors




$$\mathcal{S}_\nu^{(0)} : \quad (\tau^{(0)} + (\nu-1)\pi) - \delta < \arg z < (\tau^{(0)} + \nu\pi) + \delta, \quad \nu \in \mathbb{Z},$$

and $\exists$ unique fundamental matrices $Y_\nu(z,u)$ holomorphic in $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^0)$ having asymptotics

$$Y_\nu(z,u) \sim Y_F(z,u), \quad z \to \infty \text{ in } \mathcal{S}_\nu^{(0)}.$$

Here $\mathcal{R}(\dots)$ means universal covering of $(\dots)$.

- **Stokes matrices** $\mathbb{S}_\nu(u)$:

$$Y_{\nu+1}(z,u) = Y_\nu(z,u)\, \mathbb{S}_\nu(u).$$

Solutions at $z = 0$.

$$\exists \text{ "Levelt" form} \quad Y^{(0)}(z,u) = G^{(0)}(u)\Big(I + \sum_{j=1}^{\infty} a_j(u) z^j\Big) z^D z^L,$$

$D$ diagonal of integers, $L$ = Jordan + nilpotent.

$G^{(0)}(u)$ and $a_j(u)$ holomorphic in $\mathbb{D}(u^0)$; series uniformly convergent for $|z|$ bounded.

- **Central connection matrix $C_\nu$**: $\quad Y_\nu(z,u) = Y^{(0)}(z,u)\ C_\nu(u)$.

- Monodromy corresponding to $z \longmapsto z e^{2\pi i}$.
For $Y^{(0)}$:
$$M := e^{2\pi i L}.$$
For $Y_\nu$:
$$e^{2\pi i B}(\mathbb{S}_\nu \mathbb{S}_{\nu+1})^{-1} = C_\nu^{-1} M C_\nu.$$

- **Essential monodromy data $\mathbb{S}_0$, $\mathbb{S}_1$, $B$, $C_0$, $L$, $D$**.

System is (strongly) **isomonodromic on $\mathbb{D}(u^0)$** if the above data are constant.

Note: $\mathbb{S}_0, \mathbb{S}_1$ are enough. $\mathbb{S}_{2\nu+1} = e^{-2\pi i \nu B}\mathbb{S}_1 e^{2\pi i \nu B}, \qquad \mathbb{S}_{2\nu} = e^{-2\pi i \nu B}\mathbb{S}_0 e^{2\pi i \nu B}$.

**Theorem 1.** *System strongly isomonodromic in $\mathbb{D}(u^0)$ $\Longleftrightarrow$ $Y_\nu$, for every $\nu$, and $Y^{(0)}$, satisfy the Frobenius integrable Pfaffian system*

$$dY = \omega(z,u)Y, \qquad \omega(z,u) := \left(\Lambda(u) + \frac{A(u)}{z}\right) dz + \sum_{k=1}^{n} \Big(zE_k + \omega_k(u)\Big) du_k,$$

$$\omega_k(u) := \left(\frac{A_{ij}(\delta_{ik} - \delta_{jk})}{u_i - u_j}\right)_{i,j=1}^{3}.$$

*Equivalently, strongly isomonodromic $\Longleftrightarrow$ A satisfies the isomonodromy deformation equations*

$$dA = \sum_{j=1}^{n} \Big[\omega_k(u), A\Big] du_k.$$

**Remark.** The above theorem is analogous to the characterisation of isomonodromic deformations by Jimbo-Miwa-Ueno, including also possible resonances in $A$

## Deformations in $\mathbb{D}(u^c)$, and $u^c$ is a coalescence point

$$\frac{dY}{dz}=\left(\Lambda(u)+\frac{A(u)}{z}\right)Y,$$

$$\Delta=\mathbb{D}(u^c)\cap\left(\bigcup_{i\neq j}\{u_i-u_j=0\}\right)\neq\emptyset \quad \text{there are coalescence points.}$$

Jimbo-Miwa-Ueno theory cannot be applied.

- A fundamental matrix solution $Y(z,u)$ is holomorphic on $\mathcal{R}\Big(\mathbb{C}\backslash\{0\}\times\mathbb{D}(u^c)\backslash\Delta\Big)$, but $\Delta$ is branching locus and $Y(z,u)$ may diverge along any direction approaching $\Delta$.
- In general, monodromy data for fundamental mat. solutions $Y(z)$ of

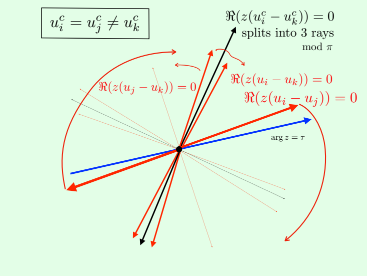
$$\frac{dY}{dz}=\left(\Lambda(u^c)+\frac{A(u^c)}{z}\right)Y, \quad \text{restricted at } u=u^c$$

  are expected to be different from those of any fundamental solution $Y(z,u)$ at point $u\notin\Delta$.
- $F_l(u)$ in formal solution have poles at $\Delta$.
- Serious problems with definition of asymptotics and Stokes sectors (see below)

- **Stokes rays of $\Lambda(u^c)$**

$$\Re((u_j^c - u_k^c)z) = 0, \quad \Im((u_j^c - u_k^c)z) < 0.$$

**Admissible direction at $u^c$** $\arg z = \tau$ not containing the above Stokes rays.

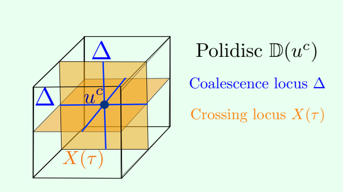



- Even if $\mathbb{D}(u^c)$ is small, as $u$ varies **some Stokes rays of $\Lambda(u)$**

$$\Re((u_j - u_k)z) = 0, \quad \Im((u_j - u_k)z) < 0.$$

cross directions $\arg z = \tau \mod \pi$.

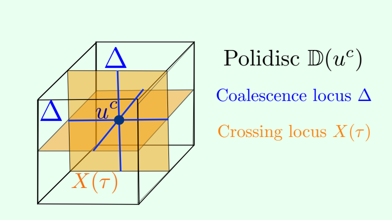
**"Crossing locus" $X(\tau)$** $= \{u \in \mathbb{D}(u^c)$ such that Stokes rays of $\Lambda(u)$ have directions $\arg z = \tau \mod \pi\}$.



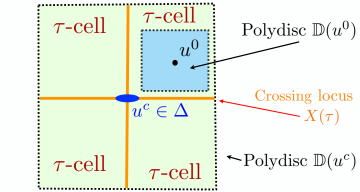

$\mathbb{D}(u^c) \backslash (\Delta \cup X(\tau))$ is not connected.

A simply connected component is a **topological cell ($\tau$-cells)**.

Anyhow, isomonodromic deformations can be defined in polydisc $\mathbb{D}(u^0)$ contained in a $\tau$-cell




$\longrightarrow$




Extension of isomonodromic deformations to the whole $\mathbb{D}(u^c)$.

**Theorem.** [Cotti, Dubrovin, DG: Duke Math. J., 168, (2019).] *Assume that:*

1. *$A(u)$ is holomorphic in $\mathbb{D}(u^c)$,*
2. *Strong isomonoromy in $\mathbb{D}(u^0)$,*
3. $A_{ij}(u) = \mathcal{O}(u_i - u_j) \longmapsto 0$ *whenever $u_i - u_j \to 0$ aproaching $\Delta$.*

*Then:*

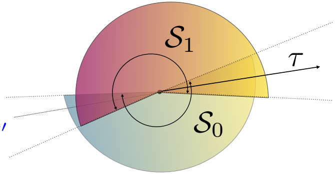
- *Fundamental matrix solutions are holomorphic in $\mathcal{R}(\mathbb{C}\backslash\{0\}) \times \mathbb{D}(u^c)$.*
  *$\Delta$ is not a branching locus.*

• *Asymptotic relations still hold on the whole $\mathbb{D}(u^c)$ in wide u-independent sectors $\mathcal{S}_\nu$*

$$Y_\nu(z,u) \sim Y_F(z,u), \quad z \to \infty, \quad u \in \mathbb{D}(u^c),$$

$$\mathcal{S}_\nu : \quad (\tau + (\nu-1)\pi) - \delta' < \arg z < (\tau + \nu\pi) + \delta'$$

$$\delta' > 0, \quad \nu \in \mathbb{Z}.$$




• *The essential monodromy data $\mathbb{S}_0$, $\mathbb{S}_1$, $B$, $C_0$, $L$, $D$ are well defined and constant on the whole $\mathbb{D}(u^c)$.*

*It suffices to compute the data for fundamental matrix solutions $Y_\nu(z) \sim Y_F(z,u^c)$ and $Y^{(0)}(z)$ of*

$$\frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y, \quad \underline{\text{restricted at } u = u^c}$$

• *Stokes matrices satisfy*

$$(\mathbb{S}_\nu)_{ij} = (\mathbb{S}_\nu)_{ji} = 0 \text{ for every } i \neq j \text{ such that } u_i^c = u_j^c.$$

□

Notice that diagonal entries $A_{ii}$ are constant.

**Proposition.**[Cotti, Dubrovin, Guzzetti (2019).] *If*

$$\underline{A_{ii} - A_{jj} \notin \mathbb{Z}\backslash\{0\}} \text{ corresponding to } u_i^c = u_j^c$$

*then* $(DS|_{u^c})$ *below only has the formal solution* $Y_F(z) = Y_F(z, u^c)$.

Thus, in order to compute the essential monodromy data of

$$(DS) \qquad \frac{dY}{dz} = (\Lambda(u) + \frac{A(u)}{z})Y$$

it suffices to compute the essential monodromy data of

$$(DS|_{u^c}) \qquad \frac{dY}{dz} = (\Lambda(u^c) + \frac{A(u^c)}{z})Y.$$

**Important conclusion:**

• **This result justifies computation of essential monodromy data on the whole $\mathbb{D}(u^c)$ starting from the system at $u^c$.**

• It gives **efficient tool for possibly explicit computations**, because $(DS|_{u^c})$ is **simpler** that $(DS)$. Indeed, $A(u^c)$ has some vanishing entries:

$$\Big(A(u^c)\Big)_{ij} = 0 \text{ whenever} u_i^c = u_j^c$$

• In order to do computations, **we just need to know $A(u^c)$**. This occurs for example in Quantum Cohomology.

Two proofs.

The one in Cotti, Dubrovin, DG (2019) is based on the study of the Stokes phenomenon and $u$-analytic continuation of fundamental solutions $Y_\nu(z,u)$ (a strategy similar to Sibuya's).

The second proof in DG: Lett. Math. Phys (2021) uses the isomonodromic Laplace transform (see also Galkin, Golyshev, Iritani: Duke Math. J (2016) for a particular case and part of the statement on analyticity of solutions).

If no parameters: $\Lambda=\Lambda(u^0)$, $A=A(u^0)$, $u^0$ fixed.

$$\frac{dY}{dz}=\left(\Lambda+\frac{A}{z}\right)Y \qquad \text{Laplace } \vec{Y}(z)\overset{\longleftrightarrow}{=}\int_\gamma e^{\lambda z}\vec{\Psi}(\lambda)d\lambda, \qquad \frac{d\Psi}{d\lambda}=\sum_{k=1}^{n}\frac{B_k}{\lambda-u_k^0}\Psi.$$

$$\text{if } \gamma \text{ is such that } e^{\lambda z}(\lambda-\Lambda)\vec{\Psi}(\lambda)\Big|_\gamma=0.$$

$$B_k=-E_k(A+I).$$

- Balser-Jurkat-Lutz. SIAM J. Math Anal. 12 (1981) (generic case, diag($A$) with no integers)
- R. Schäfke (1980-'98) • Boalch, P. (2005) • Dubrovin (1996-2004)
- DG: Funkcial. Ekvac. 59 (2016) (general case, any $A$).

We introduce deformation parameters in this picture.

A Fuchsian system

$$\frac{d\Psi}{d\lambda} = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k}\Psi, \qquad B_k(u) = -E_k(A(u)+I),$$

is strongly isomonodromic in $\mathbb{D}(u^0)$ (constant Levelt exponents, constant connection matrices $\Rightarrow$ constant monodromy matrices) if and only if it is the $\lambda$-component of a Frobenius integrable Pfaffian system

$$d\Psi = P(\lambda, u)\Psi, \qquad P(z,u) = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} d(\lambda - u_k) + \sum_{k=1}^{n} \gamma_k(u) du_k.$$

L. Schlesinger...
A.A.Bolibrukh: Izv. Akad. Nauk SSSR Ser. Mat. 41 (1997),
A.A.Bolibrukh: J. of Dynamical Control Systems, 3, (1998).

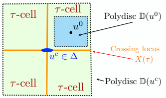
The integrability condition $dP = P \wedge P$ is the non-normalized Schlesinger system

$$\partial_i \gamma_k - \partial_k \gamma_i = \gamma_i \gamma_k - \gamma_k \gamma_i, \tag{3}$$

$$\partial_i B_k = \frac{[B_i, B_k]}{u_i - u_k} + [\gamma_i, B_k], \quad i \neq k \tag{4}$$

$$\partial_i B_i = -\sum_{k \neq i} \frac{[B_i, B_k]}{u_i - u_k} + [\gamma_i, B_i] \tag{5}$$

**Lemma** [Harnad, Boalch, DG, ...]




*The Schlesinger equations* (3)-(5) *are equivalent to the isomonodromy deformation equations*

$$dA = \sum_{j=1}^{n} [\omega_j(u), A] \, du_j$$

*of the irregular system* $\frac{dY}{dz} = \left(\Lambda(u) + \frac{A(u)}{z}\right) Y$ *if and only if*

$$\gamma_j(u) = \omega_j(u), \qquad j = 1, ..., n.$$

□

Extension to coalescences.

**Lemma** [DG] *Assume that $A(u)$ is holomorphic on the whole $\mathbb{D}(u^c)$.*

*Then, the Pfaffian system*

$$d\Psi = P(\lambda, u)\Psi, \qquad P(z,u) = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} d(\lambda - u_k) + \sum_{k=1}^{n} \gamma_k(u) du_k.$$

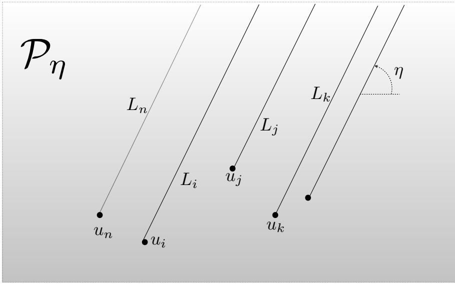
*is Frobenius integrable on the whole $\mathbb{D}(u^c)$ with holomorphic matrix coefficients*

*if and only if*

$$\left(A(u)\right)_{ij} \longrightarrow 0, \quad \Longleftrightarrow \quad [B_i(u), B_j(u)] \longrightarrow 0, \quad \text{whenever } u_i - u_j \to 0 \text{ in } \mathbb{D}(u^c).$$

For each $u \in \mathbb{D}(u^c)$, consider in $\lambda$-plane branch-cuts $L_1 = L_1(\eta)$, ..., $L_n = L_n(\eta)$ issuing from $u_1, \dots, u_n$ with direction

$$\boldsymbol{\eta := 3\pi/2 - \tau},$$




Sheet

$$\mathcal{P}_\eta(u) := \Big\{ \lambda \in \mathcal{R}(\mathbb{C}\backslash\{u_1, \dots, u_n\}) \mid \eta - 2\pi < \arg(\lambda - u_k) < \eta, \quad 1 \leq k \leq n \Big\}.$$

We define the domain

$$\mathcal{D} := \bigcup_{u \in \mathbb{D}(u^c)} \Big\{ (\lambda, u) \mid \lambda \in \mathcal{P}_\eta(u) \Big\}$$

**Note.** $\mathbb{D}(u^c)$ is "sufficiently" small...

**Theorem.**[DG: Letters in Math. Phys. (2021)]

*The Frobenius integrable*

$$d\Psi = P(\lambda,u)\Psi, \qquad P(z,u) = \sum_{k=1}^{n} \frac{B_k(u)}{\lambda - u_k} d(\lambda - u_k) + \sum_{k=1}^{n} \gamma_k(u) du_k,$$

$$B_j(u) = -E_j(A(u)+I),$$

*with*

$$\big(A(u)\big)_{ij} \longrightarrow 0, \ \textit{for } u_i - u_j \to 0 \textit{ in } \mathbb{D}(u^c).$$

*has selected vector solutions*

$$\vec{\Psi}_1(\lambda, u\,|\eta), \ \ldots\ , \ \vec{\Psi}_n(\lambda, u\,|\eta) \quad \textit{holomorphic on } \mathcal{D},$$

*and singular solutions with regular singularity at* $\lambda = u_1, \ldots, u_n$*,*

$$\vec{\Psi}_1^{(sing)}(\lambda, u\,|\eta), \ \ldots\ , \ \vec{\Psi}_n^{(sing)}(\lambda, u\,|\eta) \quad \textit{holomorphic on } \mathcal{D}.$$

They are extracted from a suitable combination of columns of a class of fundamental matrix solutions of the Pfaffian system. This class follows from results of Yoshida-Takano (1976) and Bolibruch (1977).

**Connection coefficients $c_{jk}$.**

$$\vec{\Psi}_k(\lambda, u\,|\eta) \underset{\lambda\to u_j}{=} \vec{\Psi}_j^{(sing)}(\lambda, u\,|\eta)\; c_{jk} \;+ \mathrm{reg}(\lambda - u_j), \qquad \lambda \in \mathcal{P}_\eta,$$

$c_{jk}^{(\nu)} := 0$, $\forall k = 1, \dots, n$, when $\vec{\Psi}_j^{(sing)} \equiv 0$, possibly for $A_{jj} \in -\mathbb{N} - 2$.

The $c_{jk}$'s are uniquely defined, for uniqueness of the singular behaviour of $\vec{\Psi}_j^{(sing)}$ (but $\vec{\Psi}_j^{(sing)}$ is not uniquely def. if $A_{jj} \in \mathbb{Z}_-$).

**Corollary of Theorem.** *They are* **isomonodromic connection coefficients**, *independent of $u \in \mathbb{D}(u^c)$. They satisfy the vanishing relations*

$$c_{jk} = 0 \qquad \textit{for } j \neq k \textit{ such that } u_j^c = u_k^c.$$

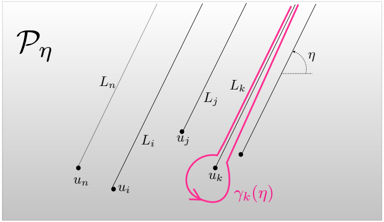
**Use selected and singular solutions in a suitable Laplace transform to re-obtain results for irregular system**

For $\nu \in \mathbb{Z}$ we define:

$$\vec{Y}_k(z,u\,|\nu) := \frac{1}{2\pi i}\int_{\gamma_k(\eta-\nu\pi)} e^{z\lambda}\vec{\Psi}_k^{(sing)}(\lambda,u\,|\eta-\nu\pi)d\lambda, \qquad \text{for } A_{kk}\notin\mathbb{Z}_-, \qquad (6)$$

$$\vec{Y}_k(z,u\,|\nu) := \int_{L_k(\eta-\nu\pi)} e^{z\lambda}\vec{\Psi}_k(\lambda,u\,|\eta-\nu\pi)d\lambda, \qquad \text{for } A_{kk}\in\mathbb{Z}_-. \qquad (7)$$

$$Y_\nu(z,u) := \left[\vec{Y}_1(z,u\,|\nu)\,\Big|\,\dots\,\Big|\,\vec{Y}_n(z,u\,|\nu)\right], \qquad \text{fixed } u\in\tau\text{-cell},\ \tau = 3\pi/2-\eta,$$



**Theorem.**[DG. Lett.Math.Phys. 2021]

*The* $Y_\nu(z,u)$*, define by Laplace transf. above, are holomorphic in* $(\lambda,u)\in\mathcal{R}(\mathbb{C}\backslash\{0\})\times\mathbb{D}(u^c)$.

*They are the fundamental matrix solutions of* $\dfrac{dY}{dz}=\left(\Lambda(u)+\dfrac{A(u)}{z}\right)Y$*, with the required asymptotics.*

*They satisfy all the properties stated in the theorem of Cotti, Dubrovin, DG ('19) mentioned before.*

*In particular, the Stokes matrices defined by*

$$Y_{\nu+1}(z,u)=Y_\nu(z,u)\mathbb{S}_\nu,$$

- *are constant in the whole* $\mathbb{D}(u^c)$,
- *the* $\mathbb{S}_\nu$ *are explicitly expressed in terms of isomonodr. connection coefficients:*

*Let*

$$\alpha_k := (e^{2\pi i A_{kk}} - 1), \ \textit{if } A_{kk} \notin \mathbb{Z}; \qquad \alpha_k := 2\pi i, \ \textit{if } A_{kk} \in \mathbb{Z},$$

*Then:*

$$(\mathbb{S}_0)_{jk} = \begin{cases} e^{2\pi i A_{kk}} \alpha_k \ c_{jk}, & j \prec k, \ u_j^c \neq u_k^c, \\ 1 & j = k, \\ 0 & j \succ k, \ u_j^c \neq u_k^c, \\ 0 & j \neq k \ , \ u_j^c = u_k^c, \end{cases}$$

$$(\mathbb{S}_1^{-1})_{jk} = \begin{cases} 0 & j \neq k, \ u_j^c = u_k^c, \\ 0 & j \prec k, \ u_j^c \neq u_k^c, \\ 1 & j = k, \\ -e^{2\pi i (A_{kk} - A_{jj})} \alpha_k \ c_{jk} & j \succ k, \ u_j^c \neq u_k^c, \end{cases}$$

$$\mathbb{S}_{2\nu+1} = e^{-2\pi i \nu B} \mathbb{S}_1 e^{2\pi i \nu B}, \quad \mathbb{S}_{2\nu} = e^{-2\pi i \nu B} \mathbb{S}_0 e^{2\pi i \nu B}$$

*Therefore*

$$(\mathbb{S}_\nu)_{jk} = (\mathbb{S}_\nu)_{kj} = 0 \qquad \textit{for } j \neq k \textit{ such that } u_j^c = u_k^c.$$

*Relation* $j \prec k$, *for* $u_j^c \neq u_k^c$, *means* $\Re(z(u_j^c - u_k^c))\Big|_{\arg z = \tau} < 0.$

See also

Sabbah C.: arXiv:2103.16878 (2021)

Sabbah's talk in November at online seminar in Kobe
(middle extension, another viewpoint for our Laplace Transform).

For the inverse problem (Riemann-Hilbert problem) in presence of coalescences, generalizing results of Malgrange: existence of an integrable deformation of a given connection at a coalescence point.

Sabbah C.: Publ. RIMS Kyoto Univ. 57, n. 3-4, (2021)

Cotti G.: Lett. Math. Phys. 111 (2021)

Cotti G.: arXiv:2105.06329 (2021)

## Example of Frobenius manifolds

Recall that canonical coordinates $u = (u_1, ..., u_n)$ coalesce (i.e. $u_i - u_j \to 0$ for some $i \neq j$) along a locus which in general has a semisimple component (Maxwell stratum) and a non semisimple one called Caustic (see Hertling's book on Frobenius and $F$ manifolds).

**Proposition:** [Cotti, Dubrovin, DG: SIGMA 16 (2020)] *If a Frobenius manifold remains semisimple at a coalescence point $u = u^c$ of the canonical coordinates $u = (u_1, ..., u_n)$, then the coefficients of the flat connection $\omega$ are such that:*

- *$A(u)$ is holomorphic at $u^c$*
- *and $A_{ij}(u^c) \to 0$.*

We can apply previous results: we can compute monodromy data on a chart from the only knowledge of $A(u^c)$.

For some important Frobenius manifolds the manifold structure is explicitly known only at coalescence points $u^c$.

**Example:** Frobenius structure on quantum cohomology of Grassmannians $QH^{\bullet}(G(k,n))$
(it is a deformation of the classical cohomology. No details, no definitions...).

We explicitly know the linear system only at the locus of small cohomology. This is a coalescence locus for almost all Grassmannians [Cotti G.: Int. Math. Res. Not. IMRN, 10.1093/imrn/rnaa163, (2020)].

We can apply our theorem on coalescences. Thus, the manifold structure can be reconstructed (in principle) from the monodromy data *computed at a coalescence point*.

- *Simplest example:* For $QH^{\bullet}(G(2,4))$. $n=6$.

$$\Lambda(u^c) = 4\sqrt{2}\cdot \mathrm{diag}(-1,-i,0,0,i,1) \qquad \longleftarrow \qquad \text{coalescence}$$

$A(u^c)$ is explicitly known and $(A(u^c))_{34} = (A(u^c))_{43} = 0$.

Computations of $S_0$, $S_1$, $C$ can be *explicitly* done. Indeed, the system at $u = u^c$ reduces to a generalised hypergeometric equation.

Up to some admissible transformations (including action of braid group) we obtain

$$S^{-1} = \begin{pmatrix} 1 & 4 & 10 & 6 & 20 & 20 \\ 0 & 1 & 4 & 4 & 16 & 20 \\ 0 & 0 & 1 & 0 & 4 & 10 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_{34} = S_{43} = 0.$$

We have also computed $C$ explicitly (too long to write here, see Cotti, Dubrovin, DG: SIGMA 16 (2020)).

Explicit computation of monodromy data allows to *verify or refine* conjectures [Dubrovin at ICM 1998, Strasburg 2013; Gamma-conjecture Galkin-Golyshev-Iritani] prescribing an explicit coincidence between the monodromy data of quantum cohomology of smooth projective varieties and suitable quantities associated with objects of exceptional collections in derived categories of coherent sheaves on these varieties.

Results of this talk allows to *justify* the theory, which is based on only knowledge at colaescence points.

If true, these conjectures would allow to obtain monodromy data in algebraic way (and then analytic continuation of Frobenius manifolds), avoiding problems of analytic computations.

Let X be a Fano manifold. The Frobenius manifold $QH^\bullet(X)$ is semisimple iff there exists a full exceptional collection $(E_1, \dots, E_n)$ in $\mathcal{D}^b(X)$. Moreover:
– the (inverse of the) Stokes matrix $\mathbb{S}$ is equal to the inverse of the Gram matrix of the Euler-Poincaré-Grothendieck product $\chi(E_i, E_j)$;
– the columns of the connection matrix $C$ coincide with the components of the forms

$$\frac{i^{\overline{d}}}{2\pi^{d/2}}\Gamma^-(X) \cup e^{-i\pi c_1(X)} \cup Ch(E_j),$$

where

$$\Gamma^-(X) = \prod_\ell \Gamma(1-\alpha_\ell), \quad \alpha'_\ell s \text{ Chern roots of } TX.$$

$d = \dim(X)$, $\overline{d} = \dim(X) \bmod 2$.

For Projective spaces [Guzzetti 1999].
For all Grassmannians [Cotti-Dubrovin-D.G arXiv '18. SIGMA '19], [Galkin-Golyshev-Iritani: Duke Math. J. '16].

**Application of "semisimple" coalescences to PVI.**

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right],$$

Consider a Frobenius integrable Pfaffian system of the type introduced before

$$dY=\omega(z,u)Y,\qquad \omega(z,u)=\left(\Lambda+\frac{A(u)}{z}\right)dz+\sum_{k=1}^{3}(zE_k+\omega_k(u))du_k,$$

$$\Lambda=\mathrm{diag}(u_1,u_2,u_3),\qquad \omega_k(u)=\left(\frac{A_{ij}(\delta_{ik}-\delta_{jk})}{u_i-u_j}\right)_{i,j=1}^{3}.$$

Suppose that:

$$\mathrm{diag}A=\mathrm{diag}(-\theta_1,-\theta_2,-\theta_3),$$
$$A \text{ has distinct eigenvalues } =0,\ \frac{\theta_\infty-\theta_1-\theta_2-\theta_3}{2},\ \frac{-\theta_\infty-\theta_1-\theta_2-\theta_3}{2}.$$

**Theorem** [Harnad (1994) $\Rightarrow$ Mazzocco (2002), Boalch (2004)] *The integrability is equivalent to PVI with parameters*

$$2\beta = -\theta_1^2, \quad 2\delta = 1 - \theta_2^2, \quad 2\gamma = \theta_3^2, \quad 2\alpha = (\theta_\infty - 1)^2.$$

**Theorem** [Mazzocco (2002); Degano & DG: arXiv:2108.07003 (2021)]. *There is a one-to-one correspondence between transcendents $y(x)$ and equivalence classes*

$$\left\{ K^0 \cdot A \cdot (K^0)^{-1}, \ K^0 = \mathrm{diag}(k_1^0, k_2^0, 1), \quad (k_1^0, k_2^0) \in \mathbb{C}^2 \backslash \{0, 0\} \right\}$$

*of solutions of isomonodromic deformation equations, with explicit formulae*

$$A(u) = (u_3 - u_1)^{\Theta} \ \Omega(x) \ (u_3 - u_1)^{-\Theta}, \quad \Theta := \mathrm{diag}(\theta_1, \theta_2, \theta_3), \quad x = \frac{u_2 - u_1}{u_3 - u_1}.$$

$$\Omega_{12} = \frac{k_1(x)}{k_2(x)} \cdot \frac{(x^2-x)\frac{dy}{dx} + (\theta_\infty - 1)y^2 + \Big(\theta_2 - \theta_1 + 1 - (\theta_\infty + \theta_2)x\Big)y + \theta_1 x}{2(x-1)y},$$

$$\Omega_{21} = \frac{k_2(x)}{k_1(x)} \cdot \frac{(x^2-x)\frac{dy}{dx} + (\theta_\infty - 1)y^2 + \Big(\theta_1 - \theta_2 + 1 - (\theta_\infty - \theta_2)x\Big)y - \theta_1 x}{2(x-y)},$$

$$\Omega_{13} = k_1(x) \cdot \frac{(x-x^2)\frac{dy}{dx} + (1-\theta_\infty)y^2 + \Big((\theta_1 - \theta_3)x + \theta_\infty + \theta_3 - 1\Big)y - \theta_1 x}{2(x-1)y},$$

$$\Omega_{31} = \frac{1}{k_1(x)} \cdot \frac{(x-x^2)\frac{dy}{dx} + (1-\theta_\infty)y^2 + \Big((\theta_3 - \theta_1)x + \theta_\infty - \theta_3 - 1\Big)y + \theta_1 x}{2x(y-1)},$$

$$\Omega_{23} = k_2(x) \cdot \frac{(x-x^2)\frac{dy}{dx} + (1-\theta_\infty)y^2 + \Big((\theta_\infty - \theta_2)x + \theta_\infty + \theta_3 - 1\Big)y - x(\theta_\infty - \theta_2 + \theta_3)}{2(x-y)},$$

$$\Omega_{32} = \frac{1}{k_2(x)} \cdot \frac{(x-x^2)\frac{dy}{dx} + (1-\theta_\infty)y^2 + ((\theta_\infty + \theta_2)x + \theta_\infty - \theta_3 - 1)y - x(\theta_\infty + \theta_2 - \theta_3)}{2x(1-y)},$$

and $\mathrm{diag}\ \Omega = \mathrm{diag}\ V = \mathrm{diag}(-\theta_1, -\theta_2, -\theta_3)$. The functions $k_j(x)$ are obtained by quadratures....

Coalescence $u_i - u_j \to 0 \Longleftrightarrow$ Singularities of PVI $x \to 0, 1, \infty$.

PVI is "traditionally" seen as the isomonodromy deformation equations (*Schlesinger equations*), of

$$\frac{d\Phi}{d\lambda} = \sum_{k=1}^{3} \frac{\mathcal{A}_k(u)}{\lambda - u_k}\Phi, \quad \underline{2 \times 2 \text{ isomonodromic Fuchsian}}, \qquad (***)$$

$$\text{eigenvalues of } \mathcal{A}_k = \pm\frac{\theta_k}{2}, \qquad \sum_{k=1}^{3} \mathcal{A}_k = \begin{pmatrix} -\theta_\infty/2 & 0 \\ 0 & \theta_\infty/2 \end{pmatrix}$$

**Solving PVI....**

The integration constants parameterizing the three singular behaviours at $x = 0, 1, \infty$ of a PVI-transcendent are funcions of *the same* traces

$$p_{jk} = \text{tr}(\mathcal{M}_j \mathcal{M}_k), \quad 1 \leq j \neq k \leq 3,$$

where $p_{jk} = p_{kj}$ and $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in SL(2, \mathbb{C})$ are the monodromy matrices of a fundamental matrix solution of $(***)$.

$\longrightarrow$ Solution of non-linear connection problem of PVI [Jimbo (1982), and several other works afterwards] .

Recall:

$$(*) \quad \frac{dY}{dz} = \left(\Lambda + \frac{A}{z}\right) Y \quad \overset{\longleftrightarrow}{\text{Laplace}} \quad \vec{Y}(z) = \int_\gamma e^{\lambda z} \vec{\Psi}(\lambda) d\lambda, \quad \frac{d\Psi}{d\lambda} = \sum_{k=1}^{n} \frac{B_k}{\lambda - u_k^0} \Psi \quad (**)$$

Since $B_k = -E_k(A+I)$, the $3 \times 3$ Fuchsian system $(**)$ can be reduced to a $2 \times 2$ Fuchsian system

$$\frac{d\Phi}{d\lambda} = \sum_{k=1}^{3} \frac{\mathcal{A}_k(u)}{\lambda - u_k} \Phi \quad (***)$$

associated with PVI (modulo a gauge transformation).

**Theorem** [Degano & DG: arXiv:2108.07003 (2021)]. *The traces $p_{jk} = tr(\mathcal{M}_j \mathcal{M}_k)$ for $(***)$ are expressed in terms of the Stokes matrices of the $3 \times 3$ irregular system $(*)$:*

$$p_{jk} = \begin{cases} 2\cos\pi(\theta_j - \theta_k) - e^{i\pi(\theta_j - \theta_k)} (\mathbb{S}_0)_{jk} (\mathbb{S}_1^{-1})_{kj}, & j \prec k, \\ 2\cos\pi(\theta_j - \theta_k) - e^{i\pi(\theta_k - \theta_j)} (\mathbb{S}_0)_{kj} (\mathbb{S}_1^{-1})_{jk},, & j \succ k. \end{cases}$$

*This holds also in case of coalescences:*

$$p_{jk} = 2\cos\pi(\theta_j - \theta_k) \quad \text{for } j \neq k \text{ such that } u_j^c = u_k^c.$$

Recall: Coalescence $u_i - u_j \to 0 \iff$ Singularities of PVI $x \to 0, 1, \infty$

(For example $u_2 - u_1 \to 0$ is $x \to 0$)

In Degano & DG: arXiv:2108.07003, we classify branches of transcendents such that

$A_{ij}(u) = \mathcal{O}(u_i - u_j) \longmapsto 0$ holomorphically, whenever $u_i - u_j \to 0$ approaching $\Delta$.

They are a sub-class of the class of transcendents with a holomorphic branch at a singular point $x = 0$, or $1$ or $\infty$ of PVI.

$\Longrightarrow$ The results of Cotti, Dubrovin, DG: Duke Math. J., 168, (2019) on coalescence can be applied, so that we can explicitly compute the associated Stokes matrices $\mathbb{S}_0, \mathbb{S}_1$ only using

$$\frac{dY}{dz} = \left(\Lambda(u^c) + \frac{A(u^c)}{z}\right) Y \qquad \text{at } u = u^c.$$

Consequently, we can compute the $p_{jk} = \mathrm{tr}(\mathcal{M}_j \mathcal{M}_k)$.

In particular: computation of the monodormy data parametrizing the chamber of a 3-dim Dubrovin-Frobenius manifold associated with a branch of a transcendent holomorphic at a singular point of PVI.

# Thank you !