Painleve equations on weighted projective spaces

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Jan/13/2021

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- Introduction
- Newton diagram and weight
- Painleve eq on the weight projective space
  - Painleve property
  - The space of initial conditions
  - K-exponent and the extended Painleve test
- Relation with singularity theory

An ODE on the complex plane.

$$\frac{dy}{dz} = f(z, y), \quad y \in \mathbf{C}^n, z \in \mathbf{C}.$$

A study of singularities of solutions.

fixed sing. ➡ sing of f

movable sing. ➡ sing depending on initial condition.

Painleve property:

ODE is said to have the Painleve property if any movable singularities are poles.

Ex. $y' = y^2 \quad \Rightarrow \quad y = -(z - c)^{-1}.$

$y' = y^3 \quad \Rightarrow \quad y = (-2z - c)^{-1/2}.$

Thm. (Poincare, Fuchs)

If a first order ODE has the Painleve property, it is equivalent to one of

(i) Solvable.

(ii) Riccati.

(iii) Weierstrass.

$$\mathrm{R} : \frac{dy}{dz} = a(z)y^2 + b(z)y + c(z).$$

$$\mathrm{W} : \left(\frac{dy}{dz}\right)^2 = 4y^3 - g_2 y - g_3.$$

Thm. (Painleve, Gambier, 1900)

If a second order ODE has the Painleve property, it is equivalent to one of

(i) Solvable.

(ii) Linear.

(iii) Weierstrass.

(iv) the Painleve equations P1 to P6.

$$\mathrm{P_I}: \frac{d^2y}{dz^2} = 6y^2 + z.$$

$$\mathrm{P_{II}}: \frac{d^2y}{dz^2} = 2y^3 + zy + \alpha.$$

$$\mathrm{P_{III}}: \frac{d^2y}{dz^2} = \frac{1}{y}\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$\mathrm{P_{IV}} : \frac{d^2y}{dz^2} = \frac{1}{2y}\left(\frac{dy}{dz}\right)^2 + \frac{3}{2}y^3 + 4zy^2 + 2(z^2-\alpha)y + \frac{\beta}{y}$$

$$\mathrm{P_{V}} : \frac{d^2y}{dz^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{z} + \delta\frac{y(y+1)}{y-1}$$

$$\mathrm{P_{VI}} : \frac{d^2y}{dz^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)\left(\frac{dy}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)\frac{dy}{dz} + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{y^2} + \gamma\frac{z-1}{(y-1)^2} + \delta\frac{z(z-1)}{(y-z)^2}\right)$$

Painleve equations are written as Hamiltonian systems.

$$\frac{dx}{dz} = -\frac{\partial H_J}{\partial y}, \quad \frac{dy}{dz} = \frac{\partial H_J}{\partial x}$$

Hamiltonian functions are given by

$$H_{\mathrm{I}} = \frac{1}{2}x^2 - 2y^3 - zy$$

$$H_{\mathrm{II}} = \frac{1}{2}x^2 - \frac{1}{2}y^4 - \frac{1}{2}zy^2 - \alpha y$$

$$H_{\mathrm{IV}} = -xy^2 + x^2y - 2xyz - 2\alpha x + 2\beta y$$

$$zH_{\mathrm{III}} = x^2y^2 - xy^2 + zx + (\alpha + \beta)xy - \alpha y$$

$$zH_{\mathrm{V}} = x(x+z)y(y-1) + \alpha_2 yz - \alpha_3 xy - \alpha_1 x(y-1)$$

$$z(z-1)H_{\mathrm{VI}} = y(y-1)(y-z)x^2 + \alpha_2(\alpha_1 + \alpha_2)(y-z)$$
$$-(\alpha_4(y-1)(y-z) + \alpha_3 y(y-z) + \alpha_0 y(y-1))x$$

The space of initial conditions.

Riccati equation. Any solution is meromorphic.

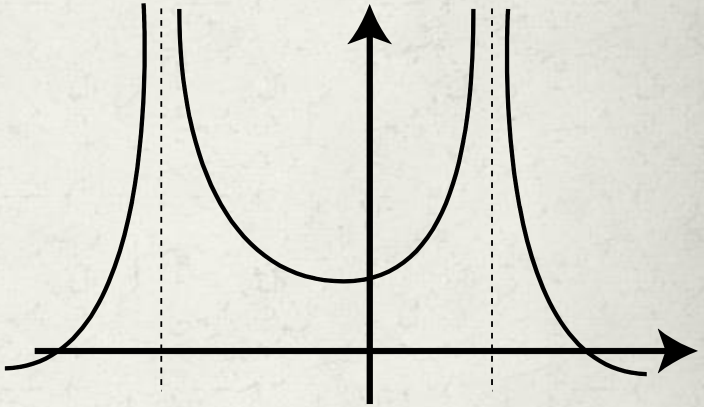
(Putting $y = u'/u$, u satisfies a linear equation.)

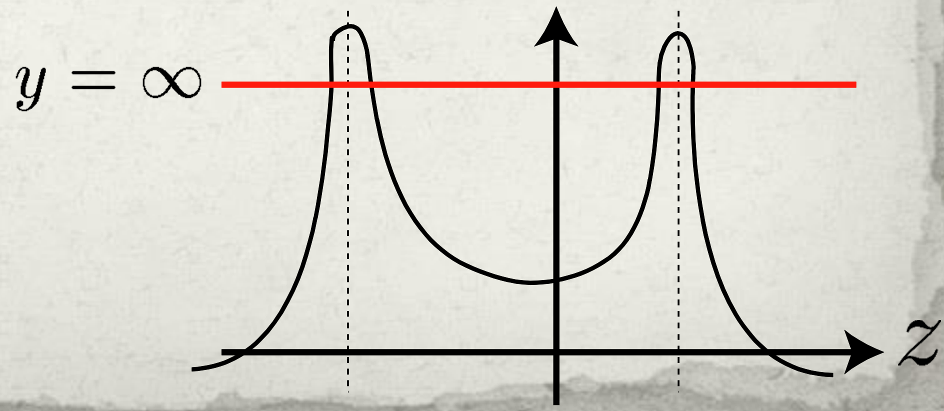
$$\frac{dy}{dz} = a(z)y^2 + b(z)y + c(z).$$

$$y = 1/\xi$$

$$\frac{d\xi}{dz} = -c(z)\xi^2 - b(z)\xi - a(z).$$

$y : \mathbf{C} \to \mathbf{C}P^1$ is holomorphic.





$\mathbf{C}P^1$ is called the **space of initial conditions**.

The space of initial conditions …

A fiber sp. of a fiber bundle, on which any solutions have analytic continuations for any z.

<u>1-dim</u>:

Riccati $\longleftrightarrow$ $\mathbf{C}P^1 \quad (g=0)$

Weierstrass $\longleftrightarrow$ torus $(g=1)$

Solvable $\longleftrightarrow$ $(g \geq 2)$

<u>2-dim</u>:

$(\mathrm{P_I}) \sim (\mathrm{P_{VI}})$ $\longleftrightarrow$ a certain class of alg. surfaces characterized by the nine points blowup of $\mathbf{C}P^2$ and the Dynkin diagrams D.

	$(\mathrm{P_I})$	$(\mathrm{P_{II}})$	$(\mathrm{P_{IV}})$	$(\mathrm{P_{III}})$	$(\mathrm{P_V})$	$(\mathrm{P_{VI}})$
D	E_8	E_7	E_6	D_6	D_5	D_4

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Newton diagram of ODEs.

Ex: The first Painleve equation.

(-1,2,1) (-1,0,2)
~~(0,2,0)~~ ~~(0,0,1)~~

$$\begin{cases} \dfrac{dx}{dz} = 6y^2 + z \\ \dfrac{dy}{dz} = x \end{cases}$$

~~(1,0,0)~~ (1,-1,1)

Newton diagram is the convex hull of these points in $\mathbf{R}^3$.
In this example, they lie on the plane

$$3x + 2y + 4z = 5$$

Newton diagram ⟷ Toric variety

In this example, the associated toric variety is the weighted projective space $\mathbf{C}P^3(3,2,4,5)$.
This space provides a suitable compactification of the natural phase space;

$$\mathbf{C}^3 = \{(x,y,z)\} \subset \mathbf{C}P^3(3,2,4,5)$$

The weighted $\mathbf{C}^*$ action:

$$(x, y, z, \varepsilon) \mapsto (\lambda^3 x, \lambda^2 y, \lambda^4 z, \lambda^5 \varepsilon). \quad \lambda \neq 0.$$

The quotient space is called the weighted projective space $\mathbf{C}_0^4/\!\sim \; = \mathbf{C}P^3(3,2,4,5)$.

A weighted projective space is an orbifold (algebraic variety) with singularities.

orbifold: $M \simeq \bigcup U_\alpha / \Gamma_\alpha$.

U_α : manifold

Γ_α : finite group

$\mathbf{C}P^3(3,2,4,5)$ is defined by $[x,y,z,\varepsilon] \sim [\lambda^3 x, \lambda^2 y, \lambda^4 z, \lambda^5 \varepsilon]$.

(i) When $x \neq 0$,

$$[x,y,z,\varepsilon] \sim [1, \frac{y}{x^{2/3}}, \frac{z}{x^{4/3}}, \frac{\varepsilon}{x^{5/3}}] := [1, Y_1, Z_1, \varepsilon_1].$$

$$\sim [1, \omega Y_1, \omega^2 Z_1, \omega \varepsilon_1].$$

The subset $\{x \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_3$.

(ii) When $y \neq 0$,

$$[x,y,z,\varepsilon] \sim [\frac{x}{y^{3/2}}, 1, \frac{z}{y^2}, \frac{\varepsilon}{y^{5/2}}] := [X_2, 1, Z_2, \varepsilon_2]$$

$$\sim [-X_2, 1, Z_2, -\varepsilon_2].$$

The subset $\{y \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_2$.

(iii) The subset $\{z \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_4$.

(iv) The subset $\{\varepsilon \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_5$.

We obtain

$$\mathbf{C}P^3(3,2,4,5) = \mathbf{C}^3/\mathbf{Z}_3 \cup \mathbf{C}^3/\mathbf{Z}_2 \cup \mathbf{C}^3/\mathbf{Z}_4 \cup \mathbf{C}^3/\mathbf{Z}_5$$

Inhomogeneous coordinates:

$$(Y_1, Z_1, \varepsilon_1), (X_2, Z_2, \varepsilon_2), (X_3, Y_3, \varepsilon_3), (X_4, Y_4, Z_4).$$

In what follows, $(X_4, Y_4, Z_4) = (x, y, z)$

$$\begin{cases} x = \varepsilon_1^{-3/5} = X_2 \varepsilon_2^{-3/5} = X_3 \varepsilon_3^{-3/5} \\ y = Y_1 \varepsilon_1^{-2/5} = \varepsilon_2^{-2/5} = Y_3 \varepsilon_3^{-2/5} \\ z = Z_1 \varepsilon_1^{-4/5} = Z_2 \varepsilon_2^{-4/5} = \varepsilon_3^{-4/5} \end{cases}$$

$$\mathbf{C}P^3(3,2,4,5) = \mathbf{C}^3/\mathbf{Z}_3 \cup \mathbf{C}^3/\mathbf{Z}_2 \cup \mathbf{C}^3/\mathbf{Z}_4 \cup \mathbf{C}^3/\mathbf{Z}_5$$

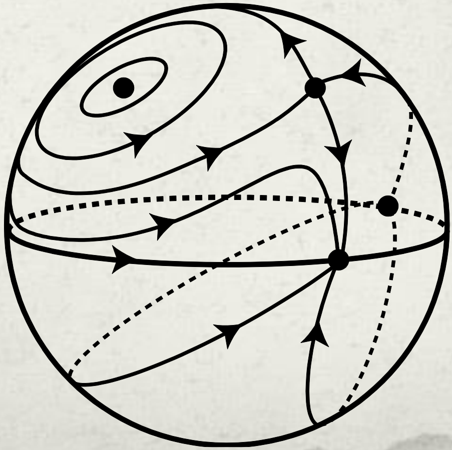
$$\begin{array}{cccc} \Cup & \Cup & \Cup & \Cup \\ (Y_1, Z_1, \varepsilon_1), & (X_2, Z_2, \varepsilon_2), & (X_3, Y_3, \varepsilon_3), & (x, y, z) \end{array}$$

Cellular decomposition

$$\mathbf{C}P^3(3,2,4,5) = \mathbf{C}^3/\mathbf{Z}_5 \cup \mathbf{C}P^2(3,2,4)$$

The first Painleve equation will be given on $\mathbf{C}^3/\mathbf{Z}_5$.
2-dim weighted proj. space $\mathbf{C}P^2(3,2,4)$ is attached at “infinity” .

A study of a singularity
($x=\infty$ or $y=\infty$ or $z=\infty$).
➡ A study of the behavior around $\mathbf{C}P^2(3,2,4)$.



$$\mathbf{C}P^3(3,2,4,5) = \mathbf{C}^3/\mathbf{Z}_3 \cup \mathbf{C}^3/\mathbf{Z}_2 \cup \mathbf{C}^3/\mathbf{Z}_4 \cup \mathbf{C}^3/\mathbf{Z}_5$$

$$\Cup \qquad \Cup \qquad \Cup \qquad \Cup$$

$$(Y_1, Z_1, \varepsilon_1),\ (X_2, Z_2, \varepsilon_2),\ (X_3, Y_3, \varepsilon_3),\ (x, y, z)$$

Give the (P1) on the fourth coord (x,y,z) .

In the other coordinates,

$$\begin{cases} \dfrac{dx}{dz} = 6y^2 + z \\ \dfrac{dy}{dz} = x \end{cases} \qquad \begin{cases} \dfrac{dY_1}{d\varepsilon_1} = \dfrac{2Y_1(2Y_1^2 + Z_1/3) - 1}{5\varepsilon_1(2Y_1^2 + Z_1/3)} \\ \dfrac{dZ_1}{d\varepsilon_1} = \dfrac{4Z_1(2Y_1^2 + Z_1/3) - \varepsilon_1}{5\varepsilon_1(2Y_1^2 + Z_1/3)}, \end{cases}$$

$$\begin{cases} \dfrac{dX_2}{d\varepsilon_2} = \dfrac{3X_2^2 - 12 - 2Z_2}{5\varepsilon_2 X_2} \\ \dfrac{dZ_2}{d\varepsilon_2} = \dfrac{4X_2 Z_2 - 2\varepsilon_2}{5\varepsilon_2 X_2}, \end{cases} \qquad \begin{cases} \dfrac{dX_3}{d\varepsilon_3} = \dfrac{4 - 4Y_3^2 + 3X_3\varepsilon_3}{5\varepsilon_3^2} \\ \dfrac{dY_3}{d\varepsilon_3} = \dfrac{-4X_3 + 2Y_3\varepsilon_3}{5\varepsilon_3^2}, \end{cases}$$

(P1) is a rational ODE on $\mathbf{C}P^3(3,2,4,5)$.

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Thm. Any solutions of (P_I) are meromorphic.

For the proof, suppose that a sol. of (P_I) has a singularity at finite $z = z_*$;

$$x(z) \to \infty \text{ or } y(z) \to \infty \text{ as } z = z_*$$

The coordinate change

$$\begin{pmatrix} X_2 \\ Z_2 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} xy^{-3/2} \\ zy^{-2} \\ y^{-5/2} \end{pmatrix}. \qquad \begin{matrix} X_2 \to 2 \\ Z_2 \to 0 \\ \varepsilon_2 \to 0 \end{matrix} \quad \text{as } z = z_*$$

It is convenient to rewrite as a 3-dim dynamical system as

$$\left\{ \begin{aligned} \frac{dX_2}{d\varepsilon_2} &= \frac{3X_2^2 - 12 - 2Z_2}{5\varepsilon_2 X_2} \\ \frac{dZ_2}{d\varepsilon_2} &= \frac{4X_2 Z_2 - 2\varepsilon_2}{5\varepsilon_2 X_2}, \end{aligned} \right. \quad \Longrightarrow \quad \left\{ \begin{aligned} \dot{X}_2 &= \tfrac{3}{2}X_2^2 - 6 - Z_2 \\ \dot{Z}_2 &= 2Z_2 X_2 - \varepsilon_2 \\ \dot{\varepsilon}_2 &= \tfrac{5}{2}\varepsilon_2 X_2 \end{aligned} \right.$$

This system has a fixed point $(X_2, Z_2, \varepsilon_2) = (2, 0, 0)$.

The solution converges to the fixed point:

$$(X_2, Z_2, \varepsilon_2) = (2, 0, 0). \qquad J = \begin{pmatrix} 6 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 5 \end{pmatrix}$$

Poincare's linearization theorem.

$$\frac{d^2y}{dz^2} = 6y^2 + z. \quad \Longrightarrow \quad \begin{cases} \dot{X}_2 = \frac{3}{2}X_2^2 - 6 - Z_2 \\ \dot{Z}_2 = 2Z_2X_2 - \varepsilon_2 \\ \dot{\varepsilon}_2 = \frac{5}{2}\varepsilon_2 X_2 \end{cases}$$

$$\frac{d^2y}{dz^2} = 6y^2.$$

$$X = X_2 - 2$$

$$\begin{cases} \dot{u} = 6u - v \\ \dot{v} = 4v - w \\ \dot{w} = 5w \end{cases} \quad \Longleftarrow \quad \begin{cases} \dot{X} = 6X - Z_2 + (\text{nonlinear}) \\ \dot{Z}_2 = 4Z_2 - \varepsilon_2 + (\text{nonlinear}) \\ \dot{\varepsilon}_2 = 5\varepsilon_2 + (\text{nonlinear}) \end{cases}$$

Linearization

Normal form theory of dynamical systems

Linearization Theorem.(Poincare)

Holomorphic vec. field on $\mathbf{C}^n$

$$Jx + f(x), \quad f \sim O(||x||^2)$$

with the fixed point $x = 0$.

If eigenvalues of the Jacobi matrix J satisfy a certain algebraic condition, then $^{\exists}$ local analytic coord. transformation near $x = 0$ s.t. $Jx + f(x)$ is transformed into the linear vec. field Jx.

Remark. Among eigenvalues 6, 4, 5 at the singularity, 4 and 5 are the same as the last two of the weight $\mathbf{C}P^3(3, 2, 4, 5)$ (not new information) .

(the eigenvalue 6) = The weighted degree of the Hamiltonian
= The Kovalevskaya exponent of (P1)
= The place of a free parameter of the Laurent series solution of (P1).

Thm. $^\exists$local analytic transformation defined near each movable singularity s.t.
(P_I) is transformed into the integrable Hamiltonian system
$y'' = 6y^2$

Cor. Any solutions of (P_I) are meromorphic.

All Painleve equations are locally transformed
into integrable equations near poles.
(necessary condition for the Painleve property)

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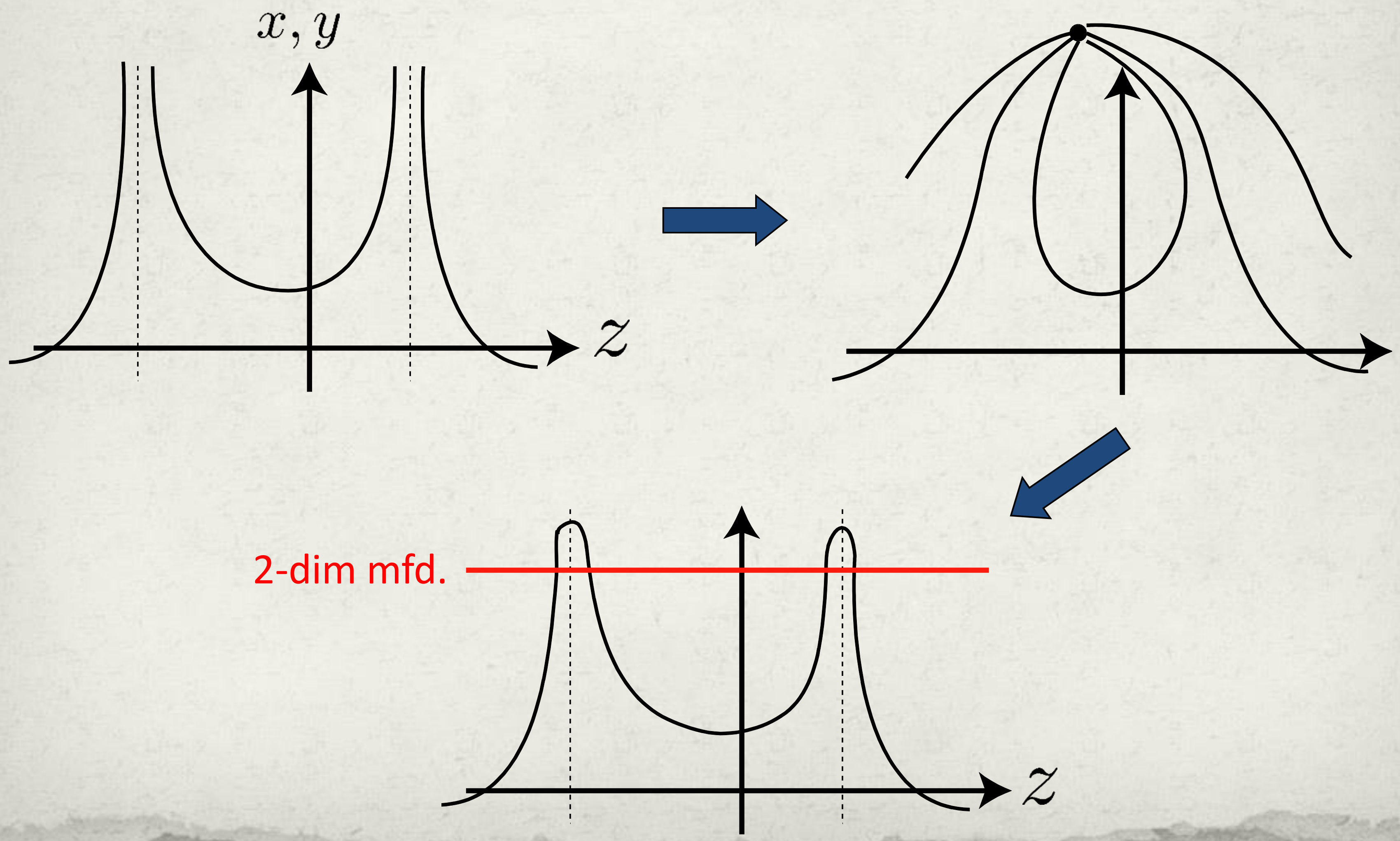
The fixed point $(X_2, Z_2, \varepsilon_2) = (2, 0, 0)$ is a singularity of the foliation defined by (P_I).

➡ resolution of sing. by a blow-up.



$$\begin{cases} \dfrac{dx}{dz} = 6y^2 + z \\ \dfrac{dy}{dz} = x \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{X}_2 = \frac{3}{2}X_2^2 - 6 - Z_2 \\ \dot{Z}_2 = 2Z_2X_2 - \varepsilon_2 \\ \dot{\varepsilon}_2 = \frac{5}{2}\varepsilon_2 X_2 \end{cases}$$

$$\overset{\text{affine}}{\Rightarrow} \quad \begin{cases} \dot{u} = 6u + (\text{nonlinear}) \\ \dot{v} = 4v + w + (\text{nonlinear}) \\ \dot{w} = 5w. \end{cases}$$

We introduce the weighted blow-up by

$$\begin{cases} u = u_1^6 = v_2^6 u_2 = w_3^6 u_3 \\ v = u_1^4 v_1 = v_2^4 = w_3^4 v_3 \\ w = u_1^5 w_1 = v_2^5 w_2 = w_3^5 \end{cases}$$

The exceptional divisor is $\mathbf{C}P^2(6,4,5)$

$$\Rightarrow \begin{cases} \dfrac{du_3}{dv_3} = \dfrac{1}{8}\left(v_3^2 w_3 + 3v_3 w_3^2 + 2w_3^3 - 8u_3 v_3 w_3^3 - 10u_3 w_3^4 + 12u_3^2 w_3^5\right) \\ \dfrac{dw_3}{dv_3} = \dfrac{1}{4}\left(4 + v_3 w_3^4 + w_3^5 - 2u_3 w_3^6\right). \end{cases}$$

$$(P_I) \Rightarrow \begin{cases} \dfrac{du}{dv} = \dfrac{1}{8}\left(v^2 w + 3vw^2 + 2w^3 - 8uvw^3 - 10uw^4 + 12u^2w^5\right) \\ \dfrac{dw}{dv} = \dfrac{1}{4}\left(4 + vw^4 + w^5 - 2uw^6\right). \end{cases}$$

The coordinate transformation is given by

$$(*) \begin{cases} x = uw^3 - 2w^{-3} - \frac{1}{2}zw - \frac{1}{2}w^2 \\ y = w^{-2} \\ z = v \end{cases}$$

- The independent value z is not transformed. $(*)$ defines a fiber bundle over z -space.
- $(*)$ is symplectic $-2du \wedge dw = dx \wedge dy$
- $\mathbf{C}^2_{(u,w)}/\mathbf{Z}_2$ is an algebraic surface $M(z)$ given by

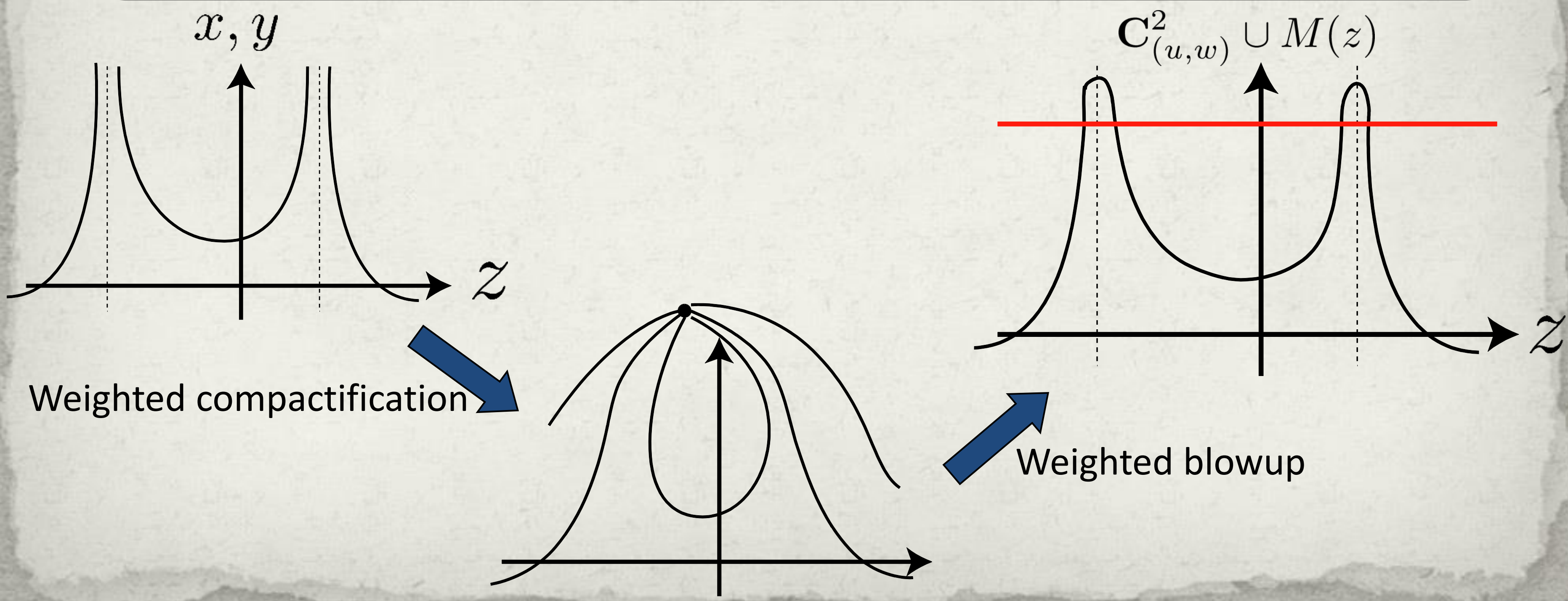
$$V^2 = UW^4 + 2zW^3 + 4W$$

- $(*)$ defines an symplectic alg. surface $\mathbf{C}^2_{(u,w)} \cup M(z)$.

Thm. The surface $\mathbf{C}^2_{(u,w)} \cup M(z)$ is a space of initial conditions for (P_I).

i.e. any solutions of (P_I) are holomorphic global sections of the fiber bundle $(\mathbf{C}^2_{(u,w)} \cup M(z)) \times \mathbf{C}_{(z)}$.

Conversely, if a given ODE is polynomial on $\mathbf{C}^2_{(u,w)} \cup M(z)$, then it is (P_I).



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Consider the n-dim polynomial system on $\mathbf{C}^n$

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z)$$

and the truncated one;

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z)$$

(A1) The truncated system is quasi-homogeneous;
$\exists$ positive integers $(p_1, \cdots, p_n, r)$ s.t.

$$f_i(\lambda^{p_1} x_1, \cdots, \lambda^{p_n} x_n, \lambda^r z) = \lambda^{1+p_i} f_i(x_1, \cdots, x_n, z)$$

Lemma. The truncated system is invariant under the $\mathbf{Z}_s$ action $(s = r+1)$,

$$(x_1, \cdots, x_m, z) \mapsto (\omega^{p_1} x_1, \cdots, \omega^{p_m} x_m, \omega^r z), \quad \omega = e^{2\pi i/s}.$$

and has a solution

$$x_i(z) = c_i (z - z_0)^{-p_i}.$$

Consider the polynomial system

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z)$$

and the truncated one;

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z)$$

(A1) $f_i(\lambda^{p_1} x_1, \cdots, \lambda^{p_n} x_n, \lambda^r z) = \lambda^{1+p_i} f_i(x_1, \cdots, x_n, z)$

(A2) $g_i(\lambda^{p_1} x_1, \cdots, \lambda^{p_n} x_n, \lambda^r z) = o(\lambda^{1+p_i}), \quad \lambda \to \infty$

(A3) The full system is also invariant under the $\mathbf{Z}_s$ action.

$$\begin{aligned}
(p_1, p_2, r) &= (2, 3, 4), \quad \text{(first Painlevé)} \\
&= (1, 2, 2), \quad \text{(second Painlevé)} \\
&= (1, 1, 1), \quad \text{(fourth Painlevé)}
\end{aligned}$$

For the system with (A1) to (A3),

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z)$$

assume the Laurent series solution

$$x_i(z) = c_i(z - z_0)^{-p_i} + a_{i,1}(z - z_0)^{-p_i+1} + a_{i,2}(z - z_0)^{-p_i+2} + \cdots$$

$\{c_i\}_{i=1}^{n}$ is a root of the equation $-p_i c_i = f_i(c_1, \cdots, c_n, 0)$.

Def. The Kovalevskaya matrix is defined by

$$K = \left\{ \frac{\partial f_i}{\partial x_j}(c_1, \cdots, c_m, 0) + p_i \delta_{ij} \right\}_{i,j=1}^{n}$$

Eigenvalues of K are called the Kovalevskaya exponents.

-1 is always the eigenvalue of K.

Laurent series solution

$$x_i(z) = c_i(z-z_0)^{-p_i} + a_{i,1}(z-z_0)^{-p_i+1} + a_{i,2}(z-z_0)^{-p_i+2} + \cdots$$

The coefficient $a_j = (a_{1,j}, \cdots, a_{m,j})^T$ satisfies

$$(K - jI)a_j = (\text{known number}).$$

Case 1. If j is not K-exp, a_j is uniquely determined.

Case 2. If j is one of the K-exp,

(2-a) no solution → no Laurent series sol.

(2-b) $\exists$solution → a_j includes an arbitrary parameter.

Classical Painleve test.

If a given n-dim ODE has the Painleve property, then there exists a leading coeffi. $\{c_i\}_{i=1}^n$ s.t. all of the associated K-exp are positive integers (except for -1).

Consider the system with (A1) to (A3).

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z)$$

The system is well-defined on the weighted proj. sp.

$M := \mathbf{C}P^{n+1}(p_1, \cdots, p_n, r, s) = \mathbf{C}^{n+1}/\mathbf{Z}_s \cup \mathbf{C}P^n(p_1, \cdots, p_n, r).$

On each inhomogeneous coordinates, rewrite it as a $n+1$ dim autonomous vector field.

We will find fixed points on the "infinity" $\mathbf{C}P^n(p_1, \cdots, p_n, r).$

Thm. The eigenvalues of the Jacobi matrix at the fixed point are given by $\lambda = r, s$ and $n-1$ Kovalevskaya exponents (except for -1).

Cor. The Kovalevskaya exponents are invariant under the action of Aut of the weighted proj. sp M.

Application: In Kawakami, Nakamura, Sakai (2018), there is a list of 4-dim Painleve equations. Among them,

$$H_{Gar}^{4+1} = p_1^2 - p_1 q_1^2 + p_2 q_1 q_2 - p_2 q_2^2 + p_1 p_2 + z p_1 - \beta q_1 + \alpha q_2$$

$$H_{\mathrm{II}}^{\mathrm{Mat}} = \frac{1}{2} p_1^2 - p_1 q_1^2 - 4 p_2 q_1 q_2 - 2 p_2 q_2^2 + p_1 p_2 + z p_1 - \beta q_1 + \alpha q_2.$$

We can conclude that they are actually different ODEs because K-exponents of them are different.

Both of them have 8 types of Laurent series. K-exp are

For H_{Gar}^{4+1},

$(4, 2, 1) \times 3$ (principle Laurent sol)

$(5, 4, -2) \times 5$ (non-principle)

For H_{II}^{Mat},

$(4, 2, 1) \times 3$

$(5, 4, -2) \times 2$

$(8, 4, -5) \times 2$

$(4, 4, -1) \times 1$

Thm. The system has n-para family of Laurent series sol,
iff there exists a singularity on the infinity set s.t.

(i) All e.values are positive integers (classical Painleve test).

(ii) The Jacobi matrix at the fixed point is semi-simple.

(iii) The system is locally linearizable via the normal form theory of dynamical systems.

If (i),(ii),(iii) hold, the singularity of the foliation is resolved by the weighted blow-up,
whose weight is given by K-exp. On the blow-up space, the system is again a polynomial system.

Conjecture. 1 to 1 correspondence:

Painleve equations $\Longleftrightarrow$ (weight) + (K-exp)

Eq.	Weight (p, q, r, s)	K-exp	h
$P_1(E_8)$	$\mathbf{C}P^3(3,2,4,5)$	6	6
$P_2(E_7)$	$\mathbf{C}P^3(2,1,2,3)$	4	4
$P_4(E_6)$	$\mathbf{C}P^3(1,1,1,2)$	3	3
$P_3(D_8)$	$\mathbf{C}P^3(-1,2,4,1)$	2	2
$P_3(D_7)$	$\mathbf{C}P^3(-1,2,3,1)$	2	2
$P_3(D_6)$	$\mathbf{C}P^3(0,1,2,1)$	2	2
$P_5(D_5)$	$\mathbf{C}P^3(1,0,1,1)$	2	2
$P_6(D_4)$	$\mathbf{C}P^3(1,0,0,1)$	2	2

h : The weighted degree of the Hamiltonian.

Eq.	Weights $(p_1, q_1, \cdot, p_n, q_n, r, s)$	K-exp	h
$(P_1)_1$	$\mathbf{C}P^3(3,2,4,5)$	6	6
$(P_1)_2$	$\mathbf{C}P^5(5,2,3,4,6,7)$	8,5,2	8
$(P_1)_3$	$\mathbf{C}P^7(7,2,5,4,3,6,8,9)$	10,7,5,4,2	10

$(P_1)_n$ is the n-th member of the first Painleve hierarchy.
(2n-dim Hamiltonian system).
Since $r = s - 1 = h - 2,$ minimal data for a weight is $(p_1, q_1, \cdots, p_n, q_n; h)$.

$$
\begin{aligned}
(p, q; h) &= (2,3;6) \quad \text{(first Painlevé)} \\
&= (1,2;4) \quad \text{(second Painlevé)} \\
&= (1,1;3) \quad \text{(fourth Painlevé)}
\end{aligned}
$$

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$(a_1, \cdots, a_n, b_1, \cdots, b_n; h)$ integers s.t. $1 \leq a_i, b_i < h$.

Characteristic function

$$\chi(T) = \frac{(T^{h-a_1}-1)(T^{h-b_1}-1)\cdots(T^{h-a_n}-1)(T^{h-b_n}-1)}{(T^{a_1}-1)(T^{b_1}-1)\cdots(T^{a_n}-1)(T^{b_n}-1)}$$

Consider the following conditions:

(B1) $\chi(T)$ is polynomial.

(B2) $a_i + b_i = h - 1$ for any i.

(B3) $\min_{1 \leq i \leq n} \{a_i, b_i\} = 1 \text{ or } 2$

<u>Lemma.</u> When n=1, the weights satisfying the condition (B) are only $(2,3;6)$, $(1,2;4)$ $(1,1;3)$. They are weights for (P_1),(P_2),(P_4), respectively.

Prop. When n=2 , the weights satisfying the condition (B) are only

$$
\begin{aligned}
(a_1, a_2, b_2, b_1; h) &= (2,3,4,5;8),\ H^{9/2}_{Gar},\quad H_{Cosgrove}, \\
&= (1,2,3,4;6),\ H^{7/2+1}_{Gar},\ H^{Mat}_{I} \\
&= (2,2,3,3;6),\ H^{(2,2,3,3)} \\
&= (1,2,2,3;5),\ H^{5}_{Gar} \\
&= (1,1,2,2;4),\ H^{4+1}_{Gar},\quad H^{Mat}_{II} \\
&= (1,1,1,1;3),\ H^{A_4}_{NY}
\end{aligned}
$$

There are corresponding 4-dim Painleve equaions.

(not unique because K-exp is not considered)

Properties of their weights and K-exp are studied in Chiba (arXiv:2010.05559).

Weight to Painleve (only 2-dim).

Step 1. Given (2,3;6), (1,2;4), (1,1;3), consider the generic quasi-homogeneous polynomials

$$H = c_1p^2 + c_2q^3,$$

$$H = c_1p^2 + c_2q^2p + c_3q^4,$$

$$H = c_1q^3 + c_2pq^2 + c_3p^2q + c_4p^3,$$

Step 2. Simplify by the symplectic transformations.

$$H = \frac{1}{2}p^2 - 2q^3,$$

$$H = \frac{1}{2}p^2 - \frac{1}{2}q^4,$$

$$H = -pq^2 + p^2q,$$

Weight to Painleve (only 2-dim).

Step 3. Versal deformation.

$$H = \frac{1}{2}p^2 - 2q^3 + \underline{\alpha_4 q} + \alpha_6,$$

$$H = \frac{1}{2}p^2 - \frac{1}{2}q^4 + \underline{\alpha_2 q^2} + \alpha_3 q + \alpha_4,$$

$$H = -pq^2 + p^2 q + \underline{\alpha_1 pq} + \alpha_2 p + \beta_2 q + \alpha_3,$$

Step 4. Replace the parameter α by z if

$$\deg(\alpha) = \deg(H) - 2$$

Result. Hamiltonians of (P1), (P2) and (P4).

$$H = \frac{1}{2}p^2 - 2q^3 + zq + \alpha_6,$$

$$H = \frac{1}{2}p^2 - \frac{1}{2}q^4 + zq^2 + \alpha_3 q + \alpha_4,$$

$$H = -pq^2 + p^2 q + zpq + \alpha_2 p + \beta_2 q + \alpha_3,$$

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