Quantum Painlevé monodromy manifolds and Sklyanin–Painlevé algebra

Marta Mazzocco

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Affine del Pezzo surfaces $\mathcal{M}_\varphi$

$$\mathcal{M}_\varphi = \{(x_1, x_2, x_3) \in \mathbb{C}^3 | \varphi(x_1, x_2, x_3) = 0\},$$
$$\varphi(x_1, x_2, x_3) = x_1 x_2 x_3 + \varphi_1(x_1) + \varphi_2(x_2) + \varphi_3(x_3).$$

Example

- Moduli space of 4-generators Sklyanin algebras up to S_3:

$$\varphi = x_1 x_2 x_3 + x_1 + x_2 + x_3.$$

- Flat deformation of $\widetilde{E}_6$ elliptic singularity [Etingof-Ginzburg]

$$\varphi = x_1 x_2 x_3 + \frac{1}{3}\left(x_1^3 + x_2^3 + x_3^3\right) + \eta_1 x_1^2 + \eta_2 x_2^2 + \eta_3 x_3^2 + \sum \omega_i x_i + \omega.$$

- SL_2–character variety of a torus with one boundary

$$\varphi = x_1 x_2 x_3 - x_1^2 - x_2^2 - x_3^2 + \omega.$$

- SL_2 character variety of a Riemann sphere with 4 holes.

$$\varphi = x_1 x_2 x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4.$$

Quantum del Pezzo surfaces

- Etingof, Oblomkov and Rains: weighted projective del Pezzo surfaces with nodal divisor.
- Etingof and Ginzburg: quantum flat deformation of cubic affine cone surfaces with an isolated elliptic singularities.
- Oblomkov: spherical sub-algebra of the rank 1 double affine Hecke algebra (DAHA).
- M.M.: spherical sub-algebras of certain degenerate DAHA.

Today:

- Quantise all affine del Pezzo of the form $\mathcal{M}_\varphi$ in such a way to produce Calabi Yau algebras.
- Introduce the generalised Painlevé–Sklyanin algebra (classical and quantum) and study its properties.

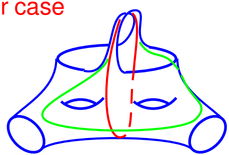
Monodromy manifolds for the Painlevé diff. equations

P-eqs	Polynomials
PVI	$x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
PV	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
PV_{deg}	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_4$
PIV	$x_1x_2x_3 - x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$
$PIII^{D_6}$	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_4$
$PIII^{D_7}$	$x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 - x_2$
$PIII^{D_8}$	$x_1x_2x_3 - x_1^2 - x_2^2 - x_2$
PII^{JM}	$x_1x_2x_3 - x_1 + \omega_2 x_2 - x_3 + \omega_4$
PII^{FN}	$x_1x_2x_3 - x_1^2 + \omega_1 x_1 - x_2 - 1$
PI	$x_1x_2x_3 - x_1 - x_2 + 1$

$$\varphi = x_1x_2x_3 - \epsilon_1 x_1^2 - \epsilon_2 x_2^2 - \epsilon_3 x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$$

Cusped character variety

Regular case



Fundamental group: $\pi_1(\Sigma_{g,s})$

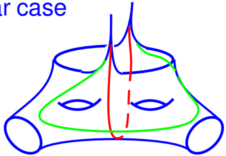
Representations:

$\mathrm{Hom}\left(\pi_1(\Sigma_{g,s}) \to SL_k(\mathbb{C})\right)$

Character variety:

$\mathrm{Hom}\left(\pi_1(\Sigma_{g,s}) \to SL_k(\mathbb{C})\right)/_{SL_k(\mathbb{C})}$

Irregular case



Fundamental groupoid of arcs: $\pi_{\mathfrak{a}}(\Sigma_{g,s,n})$

Representations:

$\mathrm{Hom}_d\left(\pi_{\mathfrak{a}}(\Sigma_{g,s,n}), SL_k(\mathbb{C})\right)$

Decorated character variety:

$\mathrm{Hom}_d\left(\pi_{\mathfrak{a}}(\Sigma_{g,s,n}), SL_k(\mathbb{C})\right)/_{\prod_{j=1}^n U_j}$

Decorated character variety

Definition

SL_2-Decorated character variety:

$$\mathrm{Hom}_{\mathfrak{d}}\left(\pi_{\mathfrak{a}}(\Sigma_{g,s,n}), SL_2(\mathbb{C})\right) /_{\prod_{j=1}^{n} U_j}$$

Lemma

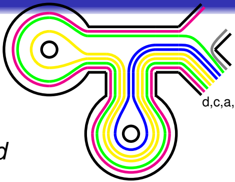
The decorated character variety is an affine variety of dimension $6g-6+3s+2n$.

Functions on the decorated character variety:

$$\begin{aligned} \mathrm{tr}_K: \quad & SL_2(\mathbb{C}) \to \mathbb{C} \\ & M \mapsto \mathrm{Tr}(MK), \qquad \text{where} \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

The coordinate ring has a cluster algebra structure and the Painlevé monodromy manifolds are special submanifolds.

Example: PV



$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + A_\infty\right) Y$$

$$\{a,b\} = ab, \quad \{a,c\} = 0, \quad \{a,d\} = -\frac{1}{2}ad$$

$$\{a,e\} = \frac{1}{2}ae, \quad \{b,c\} = 0, \quad \{b,d\} = -\frac{1}{2}bd, \quad \{b,e\} = \frac{1}{2}be,$$

$$\{c,d\} = -\frac{1}{2}cd, \quad \{c,e\} = \frac{1}{2}ce, \quad \{d,e\} = 0, \quad \{G_1,\cdot\} = \{G_2,\cdot\} = 0,$$

Monodromy manifold:

$$\mathcal{M}_V := \{x(a,b,c,d,e) |\, x, \{x,e\} = \{x,d\} = 0\}$$

$$\Rightarrow \mathcal{M}_V = \mathbb{C}[x_1,x_2,x_3]/_{\langle\varphi=0\rangle}$$

$$\begin{aligned}\varphi &= x_1x_2x_3 + x_1^2 + x_2^2 - (G_1d + G_2)x_1 - (G_2d + G_1)x_2 \\ &\quad -(d + G_1G_2)x_3 + d^2 + 1 + G_1G_2d\end{aligned}$$

Poisson structure

Poisson algebra $A_\varphi = (\mathbb{C}[x_1, x_2, x_3], \{\cdot, \cdot\}_\varphi)$ where

$$\{p, q\}_\varphi = \frac{dp \wedge dq \wedge d\varphi}{dx_1 \wedge dx_2 \wedge dx_3}$$

is the Poisson-Nambu structure on $\mathbb{C}^3$ for $p, q \in \mathbb{C}[x_1, x_2, x_3]$.

This descends to the coordinate ring of $\mathcal{M}_\varphi$:

$$\{x_1, x_2\} = \frac{\partial \varphi}{\partial x_3}, \qquad \{x_2, x_3\} = \frac{\partial \varphi}{\partial x_1}, \qquad \{x_3, x_1\} = \frac{\partial \varphi}{\partial x_2}.$$

$$\varphi = x_1 x_2 x_3 - \epsilon_1 x_1^2 - \epsilon_2 x_2^2 - \epsilon_3 x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$$

$$\{x_1, x_2\} = x_1 x_2 - 2\epsilon_3 x_3 + \omega_3, \quad \text{and cyclic,}$$

$$\{\varphi, x_i\} = 0, \qquad \forall i = 1, 2, 3.$$

For generic ω_k it is nowhere vanishing on $\mathcal{M}_\varphi$.

Universal Painlevé algebra $\mathcal{UP}$ [Chekhov-M.M.-Rubtsov 2019]

Definition

Given any scalars $\epsilon_1, \epsilon_2, \epsilon_3$, and q, $q^m \neq 1$, $\mathcal{UP}$ is the algebra with generators $X_1, X_2, X_3, \Omega_1, \Omega_2, \Omega_3$ defined by the relations:

$$\begin{aligned}
&q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 - (q^{-1}-q)\epsilon_3X_3 + (q^{-1/2}-q^{1/2})\Omega_3 = 0,\\
&q^{-1/2}X_2X_3 - q^{1/2}X_3X_2 - (q^{-1}-q)\epsilon_1X_1 + (q^{-1/2}-q^{1/2})\Omega_1 = 0,\\
&q^{-1/2}X_3X_1 - q^{1/2}X_1X_3 - (q^{-1}-q)\epsilon_2X_2 + (q^{-1/2}-q^{1/2})\Omega_2 = 0,\\
&[\Omega_i, \cdot] = 0, \qquad i = 1, 2, 3.
\end{aligned}$$

Semi-classical limit $\lim_{q\to 1} \frac{[X_i, X_j]}{1-q} = -\{x_i, x_j\}$

Example

$$\begin{aligned}
q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 = q^{-1/2}\left([X_1, X_2] + (1-q)X_2X_1\right) \quad &\Rightarrow\\
\frac{[X_1, X_2]}{1-q} + X_2X_1 - q^{1/2}(1+q)\epsilon_3X_3 + \Omega_3 \quad &\Rightarrow\\
\{x_1, x_2\} = x_1x_2 - 2\epsilon_3x_3 + \omega_3 &
\end{aligned}$$

Confluent Zhedanov algebra $\mathcal{UZ}$ [Chekhov-M.M.-Rubtsov 2019]

Definition

For any choice of three scalars Ω_i^0, $i = 1, 2, 3$, the *confluent Zhedanov algebra* $\mathcal{UZ}$ is the quotient $\mathcal{UP}/\langle \Omega_1 - \Omega_1^0, \Omega_2 - \Omega_2^0, \Omega_3 - \Omega_3^0 \rangle$.

In other words, $\mathcal{UZ}$ is s the algebra with generators X_1, X_2, X_3 defined by the relations:

$$q^{-1/2} X_1 X_2 - q^{1/2} X_2 X_1 - (q^{-1} - q)\epsilon_3 X_3 + (q^{-1/2} - q^{1/2})\Omega_3^0 = 0,$$
$$q^{-1/2} X_2 X_3 - q^{1/2} X_3 X_2 - (q^{-1} - q)\epsilon_1 X_1 + (q^{-1/2} - q^{1/2})\Omega_1^0 = 0,$$
$$q^{-1/2} X_3 X_1 - q^{1/2} X_1 X_3 - (q^{-1} - q)\epsilon_2 X_2 + (q^{-1/2} - q^{1/2})\Omega_2^0 = 0.$$

Theorem

$\mathcal{UZ}$ is a Jacobian algebra with potential Φ and central element Ω_4^0 given by:

$$\Phi = X_1 X_2 X_3 - q X_2 X_1 X_3 + \frac{q^2 - 1}{2\sqrt{q}}(\epsilon_1 X_1^2 + \epsilon_2 X_2^2 + \epsilon_3 X_3^2) + (1 - q)\sum_{k=1}^{3} \Omega_k X_k$$

$$\Omega_4^0 = \sqrt{q} X_3 X_2 X_1 - q\epsilon_1 X_1^2 - \frac{\epsilon_2}{q} X_2^2 - q\epsilon_3 X_3^2 + \sqrt{q}\Omega_1^0 X_1 + \frac{\Omega_2^0}{\sqrt{q}} X_2 + \sqrt{q}\Omega_3^0 X_3.$$

Further properties of $\mathcal{UZ}$

How about unimodularity?

Theorem

The confluent Zhedanov algebra $\mathcal{UZ}$ is a family of 3-Calabi-Yau algebras.

For $\Omega_k^0 = \epsilon_k = 0$ for all k we obtain the quantum polynomial algebra

$$\mathbb{C}\langle X_1, X_2, X_3\rangle / I, \qquad I = \langle q^{-1/2} X_1 X_2 - q^{1/2} X_2 X_1, \text{ and cyclic}\rangle.$$

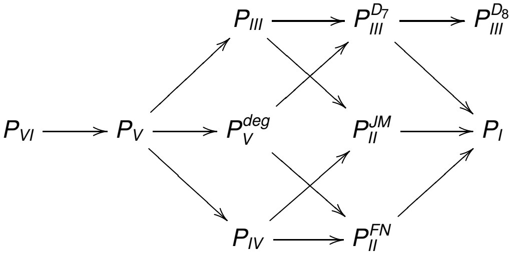
Theorem

The confluent Zhedanov algebra $\mathcal{UZ}$ is a PBW deformation of the quantum polynomial algebra.

Relation with DAHA

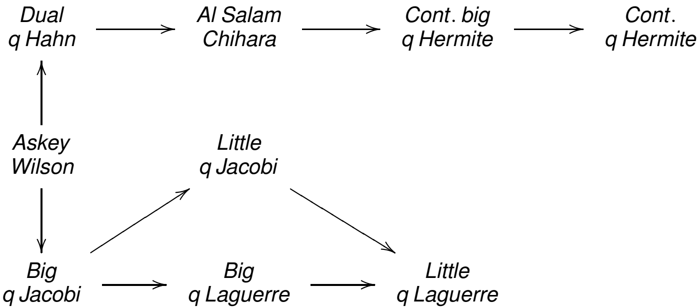
- The monodromy manifold of PVI quantises to the spherical sub–algebra of the rank 1 DAHA $\mathcal{H}$ [Oblomkov].
- Whittaker degenerations of $\mathcal{H}$ such that their spherical sub–algebra is the corresponding confluent Zhedanov algebra. [M.M. Nonlinearity '16]

Whittaker degenerations correspond to the confluence of the Painlevé differential equations:



q-Askey scheme

- All the confluent Zhedanov algebras admit representations on the space of (Laurent) polynomials.
- Elements in the *q*–Askey scheme span eigen-spaces.



Sklyanin algebra

$$Q_3(\mathcal{E}, e_0, e_1, e_2) := \mathbb{C}\langle X_1, X_2, X_3\rangle / _{\langle \frac{\partial\Phi}{\partial X_1}, \frac{\partial\Phi}{\partial X_2}, \frac{\partial\Phi}{\partial X_3}\rangle}$$

with

$$\Phi = e_0 X_1 X_2 X_3 + e_1 X_2 X_1 X_3 + \frac{e_2}{3}(X_1^3 + X_2^3 + X_3^3))$$

- For $(e_0, e_1, e_2) \in \mathcal{E}$, it is a PBW deformation of the quantum polynomial algebra [Artin-Schelter].
- It is also Calabi-Yau [Iyudu-Shkarin].

We introduce a general algebra containing all these examples as sub-cases that can be obtained by rational degenerations.

Generalised Sklyanin-Painlevé algebra [Chekhov-M.M.-Rubtsov 2019]

Definition

The *generalised Sklyanin-Painlevé algebra* is the algebra with generators X_1, X_2, X_3 and relations:

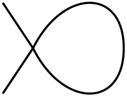
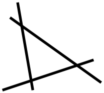
$$X_2X_3 - aX_3X_2 - \alpha X_1^2 + a_1X_1 + a_2 = 0,$$
$$X_3X_1 - bX_1X_3 - \beta X_2^2 + b_1X_2 + b_2 = 0,$$
$$X_1X_2 - cX_2X_1 - \gamma X_3^2 + c_1X_3 + c_2 = 0.$$

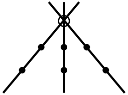
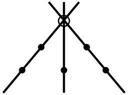
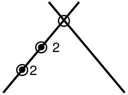
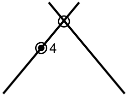
Theorem

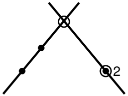
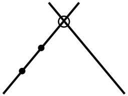
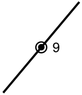
For specific choices of the parameters as follows:

1. $a = b = c \neq 0$ *and* $(a^3, \alpha\beta\gamma) \neq (-1, 1)$,
2. $(a, b, c) \neq (0, 0, 0)$ *and either* $\alpha = \beta = a - b = 0$ *or* $\gamma = \alpha = c - a = 0$ *or* $\beta = \gamma = b - c = 0$,
3. $\alpha = \beta = \gamma = 0$ *and* $(a, b, c) \neq (0, 0, 0)$,

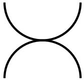
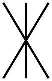
it is potential, CY, PHS and Koszul.

DAHA	Center φ for $q=1$	del Pezzo divisor D_∞	Gen. Halphen surface	Halphen divisor Δ
Elliptic $\widetilde{E}_6$	$x_1x_2x_3+x_1^3+x_2^3+x_3^3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3+$ $+x_1^3+x_2^3+x_3^3$	$A_0^{(1)}$	
GDAHA $E_6^{(1)}$	$x_1x_2x_3+x_1^3+x_2^3+$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3+$ $+x_1^3+x_2^3$	$A_0^{(1)*}$	
Deg. GDAHA	$x_1x_2x_3+x_1^3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3+x_1^3$	$A_1^{(1)}$	
	$x_1x_2x_3$ $+a_1x_1^2+b_1x_2^2+c_1x_3^2+$ $+a_2x_1+b_2x_2+c_2x_3+d$	$x_1x_2x_3$	$A_2^{(1)}$	

DAHA	Center φ for $q=1$	del Pezzo divisor D_∞	Gen. Halphen surface	Halphen divisor Δ
DAHA $\check{C}C_1$	$x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 +$ $+\omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4$	$x_1x_2x_3$	$D_4^{(1)}$	
Deg. DAHA	$x_1x_2x_3 - x_1^2 - x_2^2 +$ $\omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4$	$x_1x_2x_3$	$D_5^{(1)}$	
	$x_1x_2x_3 - x_1^2 - x_2^2 +$ $+\omega_1x_1 - x_2$	$x_1x_2x_3$	$D_7^{(1)}$	
	$x_1x_2x_3 - x_1^2 - x_2^2 - x_2$	$x_1x_2x_3$	$D_8^{(1)}$	

DAHA	Center φ for $q=1$	del Pezzo divisor D_∞	Gen. Halphen surface	Halphen divisor Δ
	$x_1x_2x_3 - x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$	$x_1x_2x_3$	$E_6^{(1)}$	
	$x_1x_2x_3 - x_1^2 + \omega_1 x_1 - x_2 - 1$	$x_1x_2x_3$	$E_7^{(1)}$	
	$x_1x_2x_3 - x_1 - x_2 + 1$	$x_1x_2x_3$	$E_8^{(1)}$	

Additive Painlevé equations

Polynomial φ	Quantum relations	Halphen surface	Divisor Δ
$x_1^3 - x_2^2 x_3$	$x_1^2 = x_2^2 = 0$ $x_3 x_2 + x_2 x_3 = 0$	$A_0^{(1)**}$	
$x_2^2 x_3 - x_1^2 x_2$	$x_2 x_3 + x_3 x_2 - x_1^2 = 0$ $x_2^2 = x_2 x_1 + x_1 x_2 = 0$	$A_1^{(1)*}$	
$x_1^3 + x_3^3$	$x_1^2 = x_2^2 = 0$	$A_2^{(1)*}$	

Outlook

- Case of degree $d > 3$ only partially understood

Polynomials φ	δ weights	φ_∞
$\frac{x_1^6}{6} + \frac{x_2^3}{3} + \frac{x_3^2}{2} + \tau x_1 x_2 x_3 + \eta_5 x_1^5 + \cdots + \omega,$	1 (1, 2, 3)	$\frac{x_1^6}{6} + \frac{x_2^3}{3} + \frac{x_3^2}{2} + \tau x_1 x_2 x_3$
$\frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^2}{2} + \tau x_1 x_2 x_3 + \eta_3 x_1^3 + \cdots + \omega,$	2 (1, 1, 2)	$\frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^2}{2} + \tau x_1 x_2 x_3$
$x_1 x_2 x_3 + x_1^5 + x_2^2 + x_3^2 + \eta_4 x_1^4 + \cdots + \omega,$	1 (2, 5, 3)	$x_1 x_2 x_3 + x_1^5 + x_2^2$
$x_1 x_2 x_3 + x_1^4 + x_2^2 + x_3^2 + \eta_3 x_1^3 + \cdots + \omega,$	2 (1, 2, 1)	$x_1 x_2 x_3 + x_1^4 + x_2^2$

- Relation with DAHA not clear for first two.
- Relation with all other multiplicative discrete Painlevé equations is not clear.