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DELTA LINK HOMOTOPY FOR TWO COMPONENT LINKS, II

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Dedicated to Professor Kazuaki Kobayashi on his sixtieth birthday

ABSTRACT

In this note, we will study Δ link homotopy (or self Δ-equivalence), which is an equivalence relation of ordered and oriented link types. Previously, a necessary condition is given in the terms of Conway polynomials for two link types to be Δ link homotopic. A pair of numerical invariants δ_1 and δ_2 classifies all (ordered and oriented) prime 2-component link types with seven crossings or less up to Δ link homotopy. We will show here that for any pair of integers n_1 and n_2 there exists a 2-component link κ such that $\delta_1(\kappa) = n_1$ and $\delta_2(\kappa) = n_2$ provided that at least one of n_1 and n_2 is even.

Keywords: Delta link homotopy, Conway polynomial, Vassiliev invariant.

1. Introduction

This note is a continuation to the previous paper [4], where the first author has given a necessary condition for two link types to be transformable into each other by a finite sequence of self Δ moves. For two link diagrams K and L which differ only in one place as in Fig. 1.1, a local move between K and L is called a Δ move. Furthermore, for two links κ and λ represented by K and L, κ and λ are said to be transformed into each other by a Δ move.

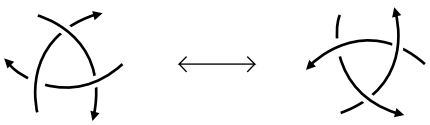


Fig. 1.1

For a μ-component link $\kappa = \kappa_1 \cup \cdots \cup \kappa_\mu$, let $\delta_1(\kappa) = a_{\mu-1}(\kappa)$, and $\delta_2(\kappa) = a_{\mu+1}(\kappa) - a_{\mu-1}(\kappa) \times (\sum_{i=1}^{\mu} a_2(\kappa_i))$, for the coefficient $a_i(\kappa)$ of the term z^i in the Conway polynomial of κ. It is known the following result by S. V. Matveev [2] and by H. Murakami and the first author [3].

Proposition 1. *Two knots (or links) can be transformed into each other by a finite sequence of Δ moves if and only if the two knots (or links) have the same number of components, and, for properly chosen orders and orientations, they have the same linking numbers between the corresponding components.*

Corollary A. *Two 2-component links can be transformed into each other by a finite sequence of Δ moves if and only if those δ_1 coincide.*

In the case that all arcs illustrated in the Fig. 1.1 are contained in the same component, the above move is called a *self Δ move* (cf. [8]). For two links κ and λ, if κ can be transformed into λ by a finite sequence of self Δ moves, κ and λ are said to be Δ *link homotopic* (or *self Δ-equivalent*). It is known the following result by the preceding work [4], which is a generalization of a work of M. Okada in [6].

Proposition 2. *If two μ-component links $\kappa = \kappa_1 \cup \cdots \cup \kappa_\mu$, $\lambda = \lambda_1 \cup \cdots \cup \lambda_\mu$ are transformed into each other by a finite sequence of self Δ moves, then we have*

$$a_{\mu+1}(\kappa) - a_{\mu-1}(\kappa) \times (\sum_{i=1}^{\mu} a_2(\kappa_i)) = a_{\mu+1}(\lambda) - a_{\mu-1}(\lambda) \times (\sum_{i=1}^{\mu} a_2(\lambda_i)).$$

Corollary B. *If two 2-component links are Δ link homotopic, then those δ_2 coincide.*

From these Corollaries A and B, we have classified all prime 2-component links with seven crossings or less in [4] (up to Δ link homotopy). There the following question is raised:

Question. Is a pair of δ_1 and δ_2 a faithful invariant of Δ link homotopy for 2-component links?

For links with 3-components or more, there is a negative example as follows: A connected sum of two Whitehead links has the Conway polynomial $\nabla(z) = z^6$ and $\delta_1 = \delta_2 = 0$. But the link is not Δ link homotopic to a trivial 3-component link. If the link should be transformed into a trivial 3-component link by a finite sequence of self Δ moves, then any sublink should be transformed into a sublink of the trivial 3-component link by a finite sequence of self Δ moves. It contradicts to the fact that a Whitehead link cannot be transformed into a trivial 2-component link by a finite sequence of self Δ moves (see [5]).

We will study the possible values of δ_1 and δ_2 as follows:

Theorem 3. (1) *For any pair of an even integer n_1 and an integer n_2, there exists a 2-component link κ such that $\delta_1(\kappa) = n_1$ and $\delta_2(\kappa) = n_2$.* (2) *For any pair of an odd integer n_1 and an even integer n_2, there exists a 2-component link κ such that $\delta_1(\kappa) = n_1$ and $\delta_2(\kappa) = n_2$.* (3) *For any pair of odd integers n_1 and n_2, there*

exists no 2-component link κ *such that* $\delta_1(\kappa) = n_1$ *and* $\delta_2(\kappa) = n_2$.

The proof of Theorem 3 will be given in Section 2. From the proof of Theorem 3, we have the following observation.

Corollary C. *If a 2-component link is invariant under a single* Δ *move, then the linking number of the link is odd.*

We remark that a single Δ move for a knot and a proper link must change the knot type and the link type (cf. [3]). The following example is observed by M. Okada [6] and by M. Teragaito [10].

Example. The 2-component link with odd linking number as in Fig. 1.2 is invariant under a Δ move. We remark that the link is invertible.

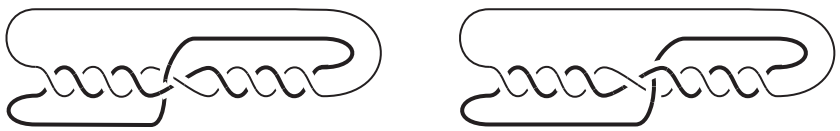


Fig. 1.2

2. Proofs of Theorem 3 and Corollary C

2.1. C_n-move and C_n-link

We use the arguments in [9] and [7]: K. Habiro introduced his C_n-move in [1] to show that two oriented knots have the same Vassiliev invariants of order less than or equal to n if and only if they are transformable into each other by a finite sequence of C_{n+1}-moves. We remark that a Δ move is Habiro's C_2-move. K. Taniyama and A. Yasuhara introduced their C_n-link in [9] to show that Habiro's C_n-move for a link κ can be realized by a band sum of the link κ and their C_n-link. Furthermore, they showed that a finite sequence of Habiro's C_{n+1}-moves can unlink, unknot, and untwist bands which connects a C_n-link and κ to be trivial bands (possibly with half-twists), and that a finite sequence of Habiro's C_{n+1}-moves can make a root of a band which connects a C_n-link and κ slide along κ and pass through a root of a band which connects another C_m-link and κ. We remark that their C_2-link is the Borromean rings. K. Taniyama, and S. Yamada, and the second author introduced their special C_n-move and special C_n-link model in [7] to show that a C_n-move is equivalent to a special C_n-move and that Habiro's C_n-move for a link κ can be realized by a band sum of the link κ and their special C_n-link model.

2.2. Δ-move

First we recall Proposition 1. If δ_1 of two 2-component links coincide, then they can be transformed into each other by a finite sequence of Δ moves. In other words,

an arbitrary 2-component link can be realized by a band sum of the $(2, 2\delta_1)$-torus link and a finite number of Borromean rings as in Fig. 2.1.

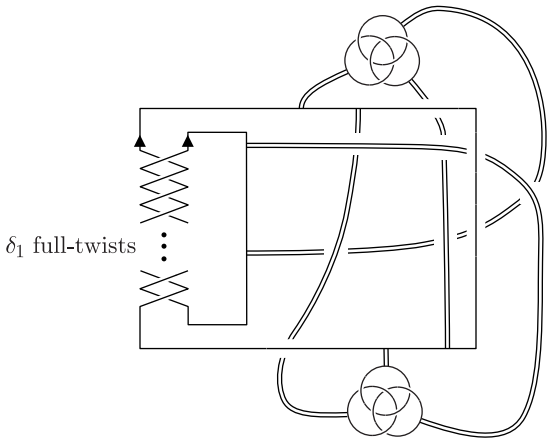


Fig. 2.1

2.3. C_3-*move*

We can arrange suitably a set of the bands and Borromean rings by using a finite sequence of Habiro's C_3-moves, and the 2-component link can be realized by a band sum of the $(2, 2\delta_1)$-torus link, a finite number of Borromean rings and C_3-links as in Fig. 2.2, where Borromean rings and their corresponding bands are suitably arranged.

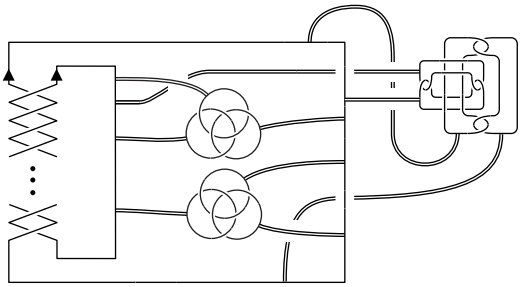
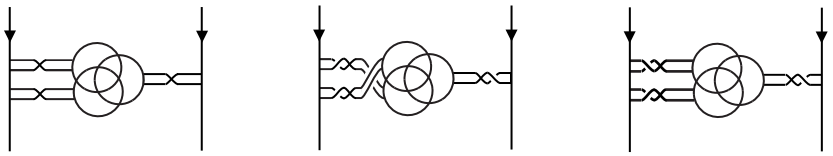


Fig. 2.2

We remark that if three roots of bands corresponding to the same Borromean rings are on the same component, that will be eliminated by a self Δ move. So we consider only the case when three roots of bands corresponding to the same Borromean rings are on distinct components. If the number of half-twisted bands corresponding to

the same Borromean rings is two or more, for example we consider that as in Fig. 2.3 (1), we perform the π-rotation by the horizontal axis to deform (1) to (2) in Fig. 2.3. The left lower band can pass over bands corresponding to another Borromean rings and C_3-links downwards along the left component to come back to the top by a finite sequence of C_3-moves as in (2) to (3). Then we can add a half-twist on each one of three bands corresponding to the Borromean rings. If a band has two half-twists, then a C_3-move eliminate it. Therefore, we can assume the number of half-twisted bands corresponding to the same Borromean rings is at most one.



(1) (2) (3)

Fig. 2.3

2.4. C_4-*move*

We can arrange suitably a set of the C_3-links and their corresponding bands by using a finite sequence of Habiro's C_4-moves, and the 2-component link can be realized by a band sum of the $(2, 2\delta_1)$-torus link, a finite number of Borromean rings, C_3-links, and C_4-links as in Fig. 2.4, where C_3-links and their corresponding bands are suitably arranged.

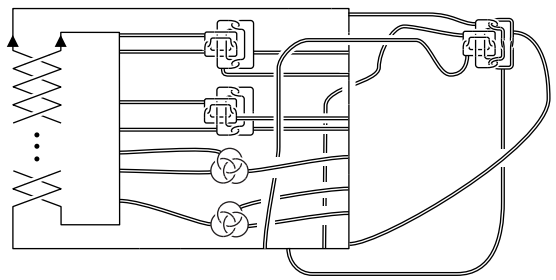
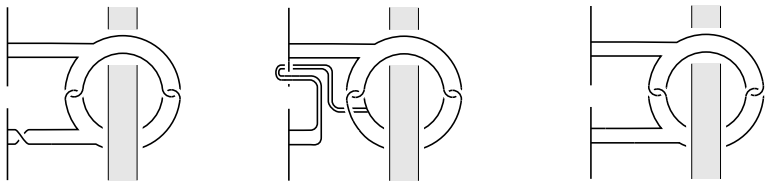


Fig. 2.4

We remark that if three of four roots of bands corresponding to the same C_3-link are on the same component, that will be eliminated by twice self Δ move. So we consider only the case when two of four roots of bands corresponding to the same C_3-link are on one component and the other two on the other component. If a band corresponding to C_3-link is half-twisted as in Fig. 2.5 (1), the band can be deformed

by an isotopy through the adjacent band into that in Fig. 2.5 (2). A C_4-move and an isotopy transform (2) into (3), where the crossings of C_3-link are changed. In the case that the adjacent band is half-twisted, the parallel argument can be applied. Therefore, we can eliminate half-twists on all bands corresponding to C_3-links by a finite sequence of C_4-moves.

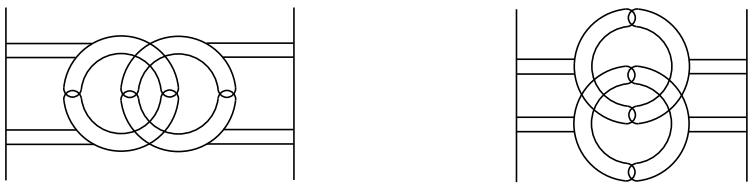


(1) (2) (3)

Fig. 2.5

2.5. *Self Δ move*

As a C_4-move is a substitution of 5-string tangles, for a C_4-move for a 2-component link, at least 3 strings of the 5 strings should be contained in the same component. Therefore, a C_4-move for a 2-component link can be realized by twice self Δ moves. Hence, a band sum of the $(2, 2\delta_1)$-torus link, a finite number of Borromean rings, C_3-links, and C_4-links as in Fig. 2.4 can be transformed into a suitably arranged band sum of the $(2, 2\delta_1)$-torus link, a finite number of Borromean rings and C_3-links by a finite sequence of self Δ moves. As stated in 2.4, it is sufficient to consider C_3-links, two roots of whose corresponding bands are on one component, and the other two on the other component. There are two types of suitably arranged band sums with C_3-links as (A) and (B) in Fig. 2.6, where the crossings of C_3-links have several possibilities.

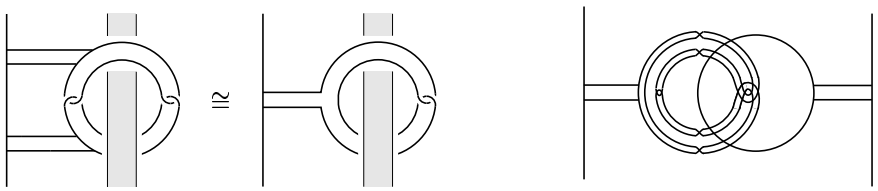


(A) (B)

Fig. 2.6

For a type (A), a suitably arranged band sum of the link and such a C_3-link can be eliminated by a finite sequence of self Δ moves. The left (or right) part of the

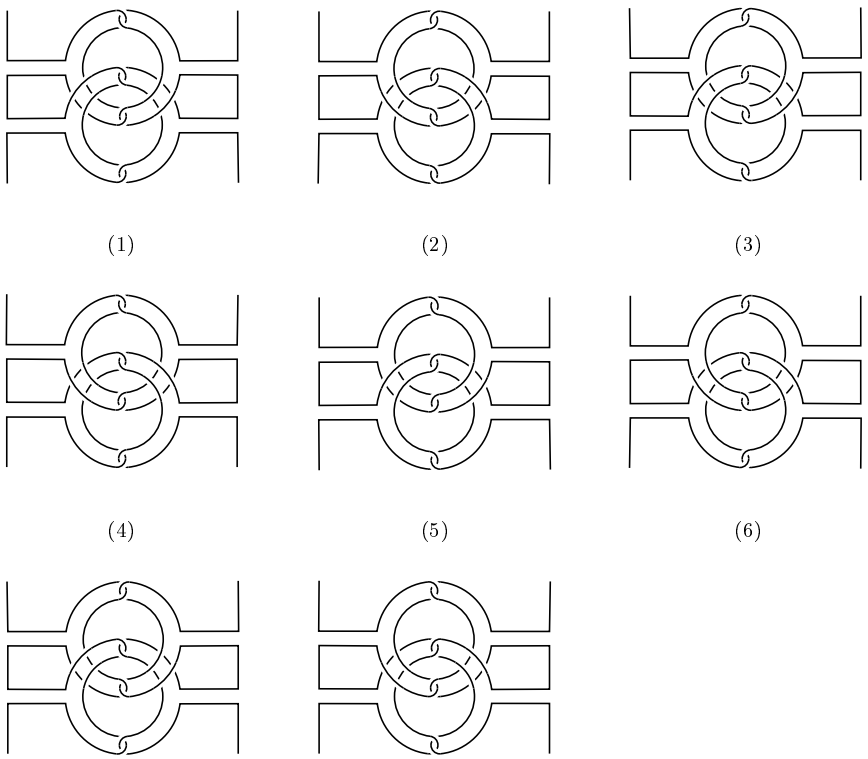
band sum can be deformed by an isotopy into a kind of Whitehead double like as in Fig. 2.7 (1). Therefore the band sum can be deformed by an isotopy into a kind of iterated Whitehead double as in Fig. 2.7 (2), which can be eliminated by a finite sequence of self Δ moves.



(1) (2)

Fig. 2.7

For a type (B), there are eight cases of crossings of C_3-links as in Fig. 2.8.

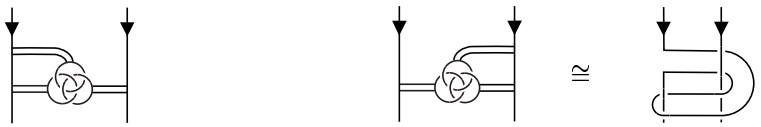


(1) (2) (3)

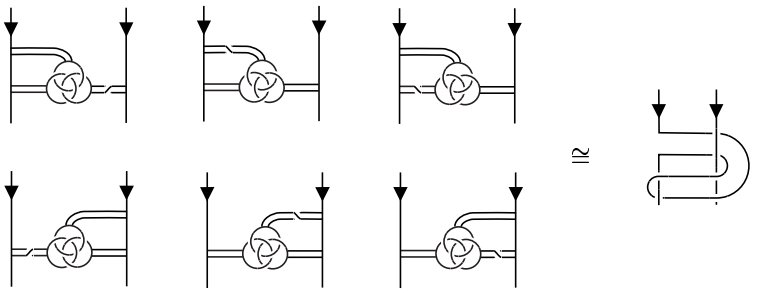
(4) (5) (6)

(7) (8)

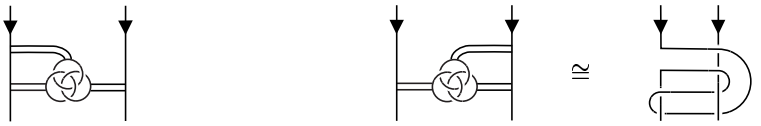
Fig. 2.8



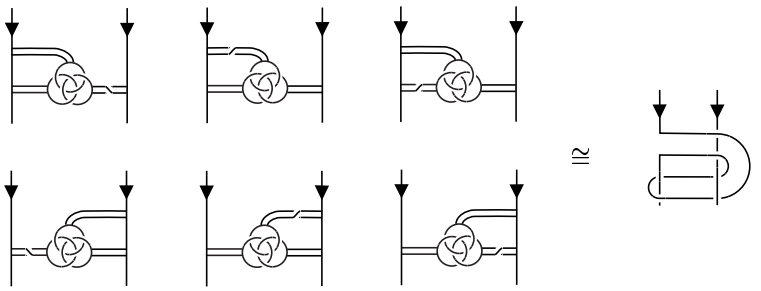
(a)



(b)



(c)



(d)

Fig. 2.9

2.6. *Whitehead grabber*

The 2-component link can be transformed into a suitably arranged band sum of the $(2, 2\delta_1)$-torus link, a finite number of Borromean rings, and C_3-links of type (B)

by a finite sequence of self Δ moves. As stated in 2.3, we can assume that three roots of bands corresponding to the same Borromean rings are on distinct components. The number of half-twisted bands corresponding to the same Borromean rings is at most one. In this case, a suitably arranged band sum of the link and Borromean rings is deformed by an isotopy into one of the following Whitehead grabbers of four types (a), (b), (c), and (d) as in Fig. 2.9.

2.7. *Effect on* δ_2

The C_3-links of type (B) in Fig. 2.8 change δ_2 of the link by ± 2, and so they keep the parity of δ_2. The Whitehead grabbers of types (a) and (b) in Fig. 2.9 change δ_2 of the link by $\pm(\delta_1 + 1)$. The Whitehead grabbers of types (c) and (d) in Fig. 2.9 change δ_2 of the link by $\pm(\delta_1 - 1)$. By combination of $\pm(\delta_1 + 1)$ and $\pm(\delta_1 - 1)$, we have the proof of Theorem 3. In the case that δ_1 is odd, δ_2 of the $(2, 2\delta_1)$-torus link is even, and both of $\pm(\delta_1 + 1)$ and $\pm(\delta_1 - 1)$ are even, and then a combination of $\pm(\delta_1+1)$ and $\pm(\delta_1-1)$ implies any even integer but no odd integer. In the case that δ_1 is even, both of $\pm(\delta_1+1)$ and $\pm(\delta_1-1)$ are odd and $\pm(\delta_1+1)-\pm(\delta_1-1) = \pm 2$, and then a combination of $\pm(\delta_1 + 1)$ and $\pm(\delta_1 - 1)$ implies an arbitrary integer. The proof of Theorem 3 is complete. □

2.8. *Proof of Corollary C*

In the case that the linking number of the link is even, i.e., δ_1 is even, a single non-self Δ move changes δ_2 by an odd integer. A single self Δ move changes the knot type of the component. Therefore, if a 2-component link is invariant under a single Δ move, then the linking number of the link is odd. The proof of Corollary C is complete. □

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