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Local Moves and Gordian Complexes, II

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Abstract. By the works of Levine [2] and Rolfsen [5], [6], it is known that a local move called a crossing-change is strongly related to the Alexander invariant. In this note, we will consider to what degree the relationship is strong. Let K be a knot, and $K^{\times}$ the set of knots obtained from a knot K by a single crossing-change. Let $\mathfrak{M}K$ be the Alexander invariant of a knot K, and $\mathfrak{M}\mathcal{K}$ the set of the Alexander invariants $\{\mathfrak{M}K\}_{K\in\mathcal{K}}$ for a set of knots $\mathcal{K}$. Our main result is the following: If both K_1 and K_2 are knots with unknotting number one, then $\mathfrak{M}K_1 = \mathfrak{M}K_2$ implies $\mathfrak{M}K_1^{\times} = \mathfrak{M}K_2^{\times}$. On the other hand, there exists a pair of knots K_1 and K_2 such that $\mathfrak{M}K_1 = \mathfrak{M}K_2$ and $\mathfrak{M}K_1^{\times} \neq \mathfrak{M}K_2^{\times}$. In other words, the Gordian complex is not homogeneous with respect to Alexander invariants.

1. Introduction

A knot K is a simple closed oriented curve in the three dimensional sphere S^3. Two knots are said to have the same knot type if there is an orientation preserving homeomorphism from S^3 to itself, which maps one knot into the other, preserving the orientation of knots. An Alexander matrix $M_K(t)$ of K is a presentation matrix of the first integral homology group $H_1(\widetilde{X_\infty})$ as a Λ-module, where $\widetilde{X_\infty}$ means the infinite cyclic covering space of the exterior X of K in S^3 and Λ means the integral group-ring $\mathbf{Z}H_1(X)$; we can see that $\mathbf{Z}H_1(X) = \Lambda$ is the Laurent polynomial ring $\mathbf{Z}[t, t^{-1}]$ where t is always taken to be represented by the meridian of K. The Λ-module $H_1(\widetilde{X_\infty})$ is said to be the Alexander invariant (or Alexander module). An Alexander polynomial $\Delta_K(t)$ of K is a generator of the order ideal of $M_K(t)$. The Alexander invariant is a stronger invariant than the Alexander polynomial; for example, the three knots $3_1\#3_1, 8_{20}, 10_{140}$ have the same Alexander polynomial $\Delta(t) = (t^2 - t + 1)^2$, and those Alexander invariants are distinct. In this note, we will give an approach to obtain more information by using Alexander invariants; for example, the two knots $5_1, 10_{132}$ have the same Alexander invariant, and they are distinguished by using Alexander invariants of their neighbourhood.

In 1937, H. Wendt [7] introduced a notion of operation for knots. We usually call the operation an unknotting operation (or briefly, a crossing-change), which

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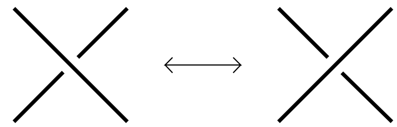

Figure 1:

is defined to be a local move between two knot diagrams K_1 and K_2 which are identical except near one point as in Fig. 1. Furthermore, we consider its spatial realization as follows: For two knots k_1 and k_2 represented by K_1 and K_2, k_1 and k_2 are said to be transformed into each other by a single crossing-change. If a knot K is transformed to a trivial knot by a single crossing-change, K is a knot with unknotting number one. M. Hirasawa and Y. Uchida [1] introduced the notion of Gordian complex by the crossing-change as follows: We consider a knot as a 0-simplex (or vertex). For a positive integer m, we consider a set of $m+1$ knots, each pair of which can be transformed into each other by a single crossing-change, as an m-simplex. We regard the set of knots as a simplicial complex, which is called the Gordian complex. They show that every 1-simplex of the Gordian complex is a face of arbitrary large dimensional simplex. In the previous note [4], the author and Y. Ohyama show that the Gordian complex is not homogeneous with respect to Alexander (Conway) polynomials, by using the difference of Alexander invariants.

The following facts are an observation on a relationship between crossing-changes and Alexander invariants. In other words, the Gordian complex is not homogeneous with respect to Alexander invariants.

Theorem 1. *If both K_1 and K_2 are knots with unknotting number one and with the same Alexander invariant, then $\mathfrak{M}K_1^{\times} = \mathfrak{M}K_2^{\times}$.*

Theorem 2. *There exists a pair of knots K_1 and K_2 such that $\mathfrak{M}K_1 = \mathfrak{M}K_2$ and $\mathfrak{M}K_1^{\times} \neq \mathfrak{M}K_2^{\times}$.*

2. Surgical description

It is well-known that any knot can be transformed into a trivial knot by crossing-changes at suitable crossing points. Every crossing-change is obtained by a ± 1 surgery along a small trivial knot around the crossing point with linking number 0. J. Levine [2] and D. Rolfsen [5], [6] introduced a surgery description of a knot and a surgical view of Alexander matrix and Alexander polynomial as follows:

Proposition 3. *Let K be a knot, K_0 a trivial knot. Then, there exist n disjoint solid tori $T_1, \cdots, T_n$ in $S^3 - K_0$ and a homeomorphism ϕ from $S^3 - {}^\circ T_1 \cup \cdots \cup {}^\circ T_n$ to itself such that*

(1) $\phi(K_0) = K$,

(2) $T_1 \cup \cdots \cup T_n$ *is a trivial link,*

(3) $\mathrm{lk}(T_i, K_0) = \mathrm{lk}(T_i, K) = 0$ *for each* i, *and*

(4) $\phi(\partial T_i) = \partial T_i$ *and* $\mathrm{lk}(\mu_i', T_i) = 1$ *where* $\mu_i \subset \partial T_i$ *is a meridian of* T_i *and* $\mu_i' = \phi^{-1}(\mu_i)$.

From a surgery description, we have a surgical view of Alexander matrix of the knot as follows:

Proposition 4. *Let* K *be a knot. Then,* K *has an Alexander matrix* $M_K(t) = (m_{ij}(t))$ *of the following form*:

(1) $m_{ij}(t) = m_{ji}(t^{-1})$, *and*

(2) $|m_{ij}(1)| = \delta_{ij}$,

where the Kronecker's delta $\delta_{ij} = 1$ (if $i = j$), $\quad 0$ (if $i \neq j$).

Here, the size of $M_K(t)$ is given by the number n in Proposition 3.

3. Proof of Theorem 1

Let $\Delta_{K_1}(t) = \Delta_{K_2}(t) = \Delta(t)$, and $k \in K_1^{\times}$. Since K_1 is a knot with unknotting number one, a surgical description of k is given by a trivial knot and a pair of solid tori. Then, from a surgical viewpoint, k has an Alexander matrix of the following form:

$$\begin{pmatrix} \Delta(t) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$$

As $r(1) = 0$ and $|m(1)| = 1, m(t^{-1}) = m(t)$, we rewrite

$$\begin{aligned} r(t) &= \pm t^r (r_1(1-t) + r_2(1-t)^2 + \cdots + r_n(1-t)^n), \text{ and} \\ m(t) &= 1 + (a_2+1)\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 + \\ & \quad \cdots + (a_{2n-2}+1)\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^{2n-2} + a_{2n}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^{2n}. \end{aligned}$$

Since K_2 is a knot with unknotting number one, a surgical description of K_2 is given by a trivial knot and a single solid torus as in the left of Fig. 2, where the solid torus is illustrated by a thick line. We transform this part of K_2 into the right of Fig. 2 by a single crossing-change, and we obtain the new knot $k^* \in K_2^{\times}$. Here, $r_1, r_2, \cdots, r_n$ mean the numbers of left-handed linkings of each part of solid torus and each parallel parts of the knot, and $a_2, a_4, \cdots, a_{2n}$ mean the numbers of left-handed full-twists of each parallel parts of the knot.

Then, from a surgical viewpoint, k^* has an Alexander matrix of the following form:

$$\begin{pmatrix} \Delta(t) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$$

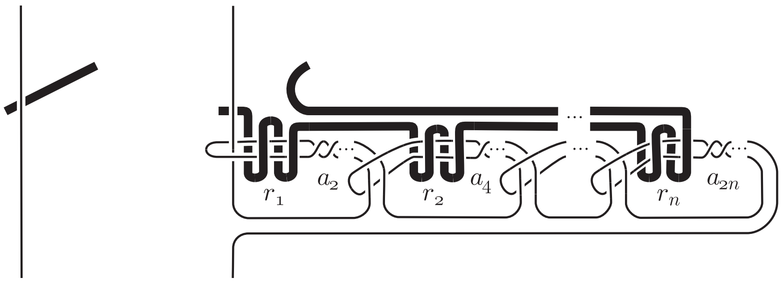

Figure 2:

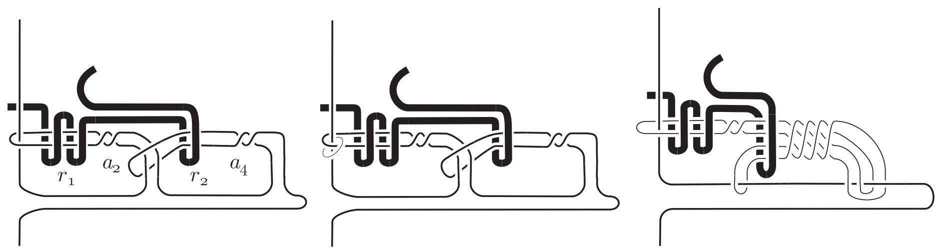

Figure 3:

We see this fact in the case $r_1 = 2, r_2 = 1, a_2 = a_4 = 1$ through surgical description as follows: We recall that k^* is obtained from K_2 by a single crossing-change. A crossing-change is realized by a ± 1 surgery along a small knot around the crossing point with linking number 0. In the center of Fig. 3, the small knot is illustrated by a thin line. By surgery along the small knot and ambient isotopy, we obtain the right of Fig. 3 from the center. From this surgical description of k^*, a part of the infinite cyclic covering space of the exterior of k^* can be seen as illustrated in Fig. 4. By reading the linking numbers between lifts of surgery knots, we can calculate $r(t)$ and $m(t)$.

Therefore, we have $\mathfrak{M}K_1^\times \subset \mathfrak{M}K_2^\times$, and vice versa. The proof is complete.

Remark. The parallel argument shows the following: *if two knots K_1, K_2 have the*

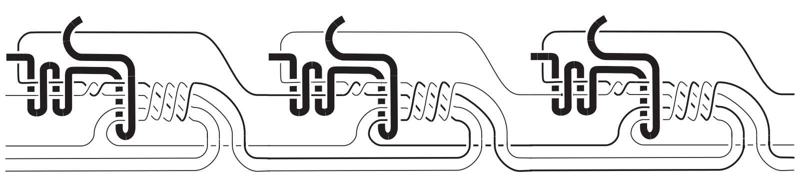

Figure 4:

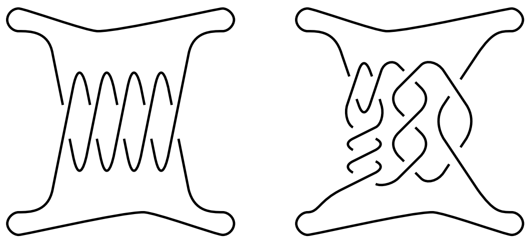

Figure 5: 5_1 and 10_{132}

same Alexander matrix from a surgical viewpoint, then $\mathfrak{M}K_1^\times = \mathfrak{M}K_2^\times$.

4. Proof of Theorem 2

We consider a pair of knots, 5_1 and 10_{132}, which have the same Alexander invariant $(t^4 - t^3 + t^2 - t + 1)$. It is easily seen that 5_1 is transformed to a trefoil knot by a single crossing-change. So we have $(t^2 - t + 1) \in \mathfrak{M}5_1^\times$.

We see that $\mathrm{u}(10_{132}) = 1$. Therefore, from a surgical viewpoint, a knot obtained from 10_{132} by a single crossing-change has an Alexander matrix of the following form:

$$\begin{pmatrix} \pm(t^2 - t + 1 - t^{-1} + t^{-2}) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$$

Here we would assume that the determinant is $\pm(t - 1 + t^{-1})$. Put $t = -1$, and and we have the determinant

$$\begin{vmatrix} \pm 5 & r(-1) \\ r(-1) & m(-1) \end{vmatrix} = \pm 3$$

From the equation modulo 5, we have $r(-1)^2 \equiv \pm 3 \mod 5$. It is a contradiction. Hence, we have $(t^2 - t + 1) \notin \mathfrak{M}10_{132}^\times$. Hence, we have $\mathfrak{M}5_1^\times \neq \mathfrak{M}10_{132}^\times$. The proof is complete.

Remark. In the proof, we can see that the pair of knots 5_1 and 10_{132} have the same Alexander invariant, and that they have the distinct surgical views of Alexander matrices.

5. Addendum

The following proposition is observed at Ikarashi. Here, $\Delta K^\times$ means the set of the Alexander polynomials $\{\Delta_k(t)\}_{k \in K^\times}$.

Proposition 5. *Let* K *be a knot with* $\mathrm{u}(K) = 1$ *and* $\Delta_K(t) = t^2 - t + 1$. *Then,* $t^2 - 3t + 1 \notin \Delta K^\times$.

Proof. If there would exist a knot k such that $\mathrm{d_G}(K, k) = 1$ and $\Delta_k(t) = t - 3 + t^{-1}$, then, from a surgical viewpoint, k has an Alexander matrix of the following form:

$$\begin{pmatrix} \pm(t-1+t^{-1}) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}$$

And the determinant should be $\pm(t-3+t^{-1})$. We have the determinant modulo t^2-t+1,

$$r(t^{-1})r(t) \equiv \pm 2 \mod (t^2-t+1).$$

We consider $r(t) \equiv at-b \mod (t^2-t+1)$, then $r(t^{-1})r(t) \equiv a^2+b^2-ab \mod (t^2-t+1)$. For a pair of integers a,b, the right hand side a^2+b^2-ab is a multiple of 4 or an odd number, and so $a^2+b^2-ab \not\equiv \pm 2 \mod (t^2-t+1)$. It is a contradiction. The proof is complete. □

Remark. In the previous note [3], we see: *Let K be a knot with* $\mathrm{u}(K)=1$ *and* $\Delta_K(t)=t^2-3t+1$. *Then,* $t^2-t+1 \notin \Delta K^{\times}$. The parallel proof will also show this observation. There are still open that: Are there a pair of knots K_1, K_2 such that $\mathrm{d_G}(K_1,K_2)=1$ and $\Delta_{K_1}(t)=t^2-t+1, \Delta_{K_2}(t)=t^2-3t+1$?

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