A system of differential equations in 4 variables of rank 5 invariant under the Weyl group of type E_6

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1 Introduction

This paper describes a detailed procedure of finding a system of differential equations in 4 variables of rank 5 invariant under the Weyl group of type E_6 that is announced in [5]. This system is the uniformizing equation of the complex hyperbolic structure of the moduli space of marked cubic surfaces found in [1]. It is defined on a Zariski open subset of $\mathbf{C}^4$, which admits a biregular action of the Weyl group of type E_6; our system is invariant under this action.

We first fix the notation and recall the complex hyperbolic structure on the moduli space of marked cubic surfaces. The moduli space of marked cubic surfaces, which we denote by M, is studied for example in [3] and [6]. An important and famous fact is that M admits a biregular action of the Weyl group of type E_6. Since any nonsingular cubic surface can be obtained by blowing up the projective plane $\mathbf{P}^2$ at six points, it can be represented by a 3×6-matrix of which columns give homogeneous coordinates of the six points. In order to get a smooth cubic surface from six points, we assume that no three points are collinear and the six points are not on a conic. On the set of 3×6 matrices, we have a canonical action of GL_3 on the left and an action of the group $\mathbf{C}^\times$ on the right. By killing such ambiguity of coordinates, for every such matrix, we get the representative in the following form:

$$x=\begin{pmatrix}1&0&0&1&1&1\\0&1&0&1&x^1&x^2\\0&0&1&1&x^3&x^4\end{pmatrix}.$$

The cubic surface obtained by blowing up the six points represented by the matrix x is non-singular if and only if the quantity

$$\begin{aligned}D(x)\ :=\ & x^1x^2x^3x^4(x^1-1)(x^2-1)(x^3-1)(x^4-1)\\ &\times(x^1-x^2)(x^1-x^3)(x^2-x^4)(x^3-x^4)\\ &\times\{x^1x^4-x^2x^3\}\{(x^1-1)(x^4-1)-(x^2-1)(x^3-1)\}\\ &\times\{x^1(x^2-1)(x^3-1)x^4-(x^1-1)x^2x^3(x^4-1)\}\end{aligned}$$

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does not vanish. Thus we can identify the moduli space M with the affine open set $\{x = (x^1, \ldots, x^4) \mid D(x) \neq 0\}$. Note that, though the action of G is biregular on M, it is birational in terms of x. The corresponding cubic surface $CS(x) \subset \mathbf{P}^3$ is defined, for example, by the following cubic equation in $t_1, \ldots, t_4$:

$$\begin{aligned} P(t,x) \quad := \quad & (vw^2 - uz^2 + uvz + uwz^2 - vuw - vzw^2)t_2t_4^2 \\ & +(-vw^2 + uz^2 + v^2w^2 - u^2z^2 - uv^2 + u^2v)t_2t_3t_4 \\ & +uv(v - z - vw - u + w + uz)(t_2t_3^2 - t_1^2t_4) \\ & -uv(uz^2 - wz^2 - vw^2 + zw^2 - uz + vw)t_2^2t_4 \\ & +(-vw + uz - uvz - uwz + vuw + vwz)(t_1t_4^2 - uvt_2^2t_3) \\ & +(-u^2v + uv^2 + u^2z - uz - wv^2 + vw)t_1t_3t_4 \\ & +(-z^2 + w^2 + uz^2 - vw^2 + v - u)uvt_1t_2t_4 \\ & +(-uw + vz - uvz + vuw - v + u)vut_1t_2t_3 = 0, \end{aligned}$$

where, for notational simplicity, we just temporally wrote $x^1 = u$, $x^2 = v$, $x^3 = w$, $x^4 = z$.

Let $TC(x)$ be the triple cover of $\mathbf{P}^3$ branching along $CS(x)$; it is a 3-fold in $\mathbf{P}^4$ given by $t_0^3 + P(t,x) = 0$. Let $\varphi := (t_0^3 + P(t,x))^{-2}t_0\Omega$ be a meromorphic 4-form on $\mathbf{P}^4$, where $\Omega = \sum_{i=0}^4 (-1)^i t_i dt_0 \wedge \cdots \overset{\vee}{dt_i} \cdots \wedge dt_4$. This 4-form φ represents an element of the primitive cohomology group $H_0^{2,1}(TC(x), \mathbf{C})$ through the Griffiths-Poincaré residue map Res:

$$\int_\gamma \mathrm{Res}\, \varphi = \int_{T(\gamma)} \varphi, \quad \gamma \in H_3(TC(x), \mathbf{Z}),$$

where $T(\gamma)$ is a tubular neighborhood of γ. Let ω be a primitive cubic root of unity, and put $\mathcal{E} = \mathbf{Z}[\omega]$. Following is known ([1], [2]):

(i) For a suitable $\mathcal{E}$-basis $\gamma_1, \ldots, \gamma_5$ of $H_3(TC(x), \mathbf{Z})$, the multi-valued period map

$$f : M \ni x \longmapsto v_1(x) : \cdots : v_5(x) \in \mathbf{P}^4,$$

where $v_i(x) = \int_{\gamma_i} \mathrm{Res}\, \varphi$, has its image in the ball

$$B^4 = \{v_1 : \cdots : v_5 \in \mathbf{P}^4 \mid h(v) := |v_1|^2 - |v_2|^2 - \cdots - |v_5|^2 > 0\}.$$

(ii) The projective monodromy group of f is the principal congruence subgroup

$$\Gamma(1-\omega) := \{g \in \Gamma \mid g \equiv I_5 \bmod (1-\omega)\}/\text{center},$$

with level $(1-\omega)$, of the modular group

$$\Gamma := \{g \in GL_5(\mathcal{E}) \mid {}^t\bar{g}hg = h\}/\text{center},$$

where h denotes the matrix representing the hermitian form above. Moreover, the isomorphism $\mathcal{E}/(1-\omega)\mathcal{E} \cong \mathbf{F}_3$ (the field with three elements) induces the isomorphisms

$$\Gamma/\Gamma(1-\omega) \cong \{g \in GL_5(\mathbf{F}_3) \mid {}^tghg = h\}/\text{center} = G,$$

which naturally acts on the quotient space $B^4/\Gamma(1-\omega)$ and is isomorphic to the Weyl group of type E_6.

(iii) $\Gamma(1-\omega)$ is a reflection group; let $\mathcal{H}$ be the union of the mirrors (in the ball) of the reflections. Then f induces the G-equivariant isomorphism

$$M \xrightarrow{\cong} (B^4 - \mathcal{H})/\Gamma(1-\omega).$$

Since the functions $v_i(x)$ are defined by the integrals given above, they should satisfy a system of differential equations defined on M of rank (= dimension of local solutions at a(ny) generic point) 5. In the following sections, we explain how we found its explicit form. This system is unknown so far; in the final section, we see that its restriction to any irreducible component of $\mathbf{C}^4 - M$ is the Appell-Lauricella hypergeometric system that is the uniformizing differential system for the configuration space of six points in the projective plane.

The recipe for finding the system is as follows: Since M is covered by the ball and f is the developing map, f is PGL_4-multi-valued. We apply the Schwarzian derivatives

$$S^k_{ij}\{f;x\} =: S^k_{ij}(x), \quad i,j,k = 1,\dots,4$$

to the map f with respect to the coordinates $x = (x^1,\dots,x^4)$. Thanks to PGL_4-invariance of the Schwarzian derivatives, $S^k_{ij}(x)$ are single-valued, and so they are rational functions with poles only along $\{D = 0\}$. The map f can be recovered (up to multiplying a function) by linearly independent solutions of the system

$$E: \quad \frac{\partial^2 v}{\partial x^i \partial x^j} = \sum_{k=1}^{4} S^k_{ij} \frac{\partial v}{\partial x^k} + S^0_{ij} v, \qquad (1 \le i,j,k \le 4)$$

where the coefficients S^0_{ij} are polynomials in S^k_{ij} and their derivatives, as given below. The local behavior and the integrability condition would determine the system E, since Mostow rigidity does not allow the existence of extra parameters. Instead of computing directly the integrability condition, we take advantage of the invariance of E under the action of G on M.

2 Systems of differential equations in $n(\geq 2)$ variables of rank $n+1$

We review basic facts about the system

$$E_n: \quad D_i D_j u = \sum_k S^k_{ij} D_k u + S^0_{ij} u \qquad (1 \le i,j,k \le n),$$

where $D_j = \partial/\partial x^j$; refer to [7]. This system is of rank at most $n+1$, and is exactly of rank $n+1$ if and only if the coefficients satisfy the following (integrability) condition:

$$S^k_{ij} = S^k_{ji}, \qquad S^0_{ij} = S^0_{ji},$$

$$D_kS^m_{ij}+\sum_t S^t_{ij}S^m_{kt}+\delta^m_kS^0_{ij}=D_iS^m_{kj}+\sum_t S^t_{kj}S^m_{it}+\delta^m_iS^0_{kj},$$

$$D_kS^0_{ij}+\sum_t S^t_{ij}S^0_{kt}=D_iS^0_{kj}+\sum_t S^t_{kj}S^0_{it},$$

where δ denotes Kronecker's symbol. In particular, the coefficients S^0_{ij} can be expressed as polynomials in S^k_{ij} and their derivatives:

$$S^0_{ij}=-D_kS^k_{ij}-\sum_t S^t_{ij}S^k_{kt}+D_iS^k_{kj}+\sum_t S^t_{kj}S^k_{it},$$

where $k\neq i$ (recall the assumption $n\geq 2$). When these conditions are satisfied, the system E_n is called an *integrable* system.

Put

$$S_i:=\sum_j S^j_{ij},\qquad (1\leq i\leq n).$$

By replacing the unknown u by its product with a non-zero function of x, any integrable system of the form E_n can be transformed uniquely into a system satisfying an additional condition: $S_1=\cdots=S_n=0$. The coefficients are transformed as follows

$$\begin{aligned} S^k_{ij} &\longrightarrow S^k_{ij}-\frac{1}{n+1}(\delta^k_iS_j-\delta^k_jS_i),\\ S^0_{ij} &\longrightarrow S^0_{ij}(D_jS_i+\sum_h S_hS^h_{ij})-(\frac{1}{n+1})^2S_iS_j. \end{aligned}$$

Such systems will be called *normal*, and can be characterized as systems such that the Wronskian of any linearly independent solutions is constant. Since we are interested only in the ratio of linearly independent solutions, we can always assume that our system is normal.

For a non-degenerate map (Jacobian $\not\equiv 0$) $x=(x^1,\ldots,x^n)\mapsto z=(z^1,\ldots,z^n)$, the Schwarzian derivatives are defined as

$$S^k_{ij}\{z;x\}=\sum_p\frac{\partial^2z^p}{\partial x^i\partial x^j}\frac{\partial x^k}{\partial z^p}-\frac{1}{n+1}\left(\delta^k_i\sum_{p,q}\frac{\partial^2z^p}{\partial x^q\partial x^j}\frac{\partial x^q}{\partial z^p}+\delta^k_j\sum_{p,q}\frac{\partial^2z^p}{\partial x^q\partial x^i}\frac{\partial x^q}{\partial z^p}\right),$$

$1\leq i,j,k\leq n$. They have the properties

(1) (projective invariance)

$$S^k_{ij}\{Az;x\}=S^k_{ij}\{z;x\}\quad\text{for}\quad A\in PGL_{n+1}.$$

(2) (connection formula) For a change of coordinates from x to y,

$$S^k_{ij}\{z;y\}=S^k_{ij}\{x;y\}+\sum_{p,q,r}S^r_{pq}\{z;x\}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r}.$$

(3) (local behavior along ramifying singularities) If $z=z(x)$ is ramified along $\{x^1=0\}$ with exponent α, that is,

$$z^1(x)=(x^1)^\alpha v^1,\quad z^2(x)=v^2,\ldots,z^n(x)=v^n,\quad \det\left(\frac{\partial z^i}{\partial x^j}\right)=(x^1)^{\alpha-1}u,$$

where $v^j(1 \leq j \leq n)$ and u are holomorphic functions not divisible by x^1, then

$$S^k_{ij}\{z;x\}, \quad S^k_{1j}\{z;x\} + \delta^k_j \tfrac{1}{n+1}\tfrac{\alpha-1}{x^1}, \quad (x^1)^{-1}S^1_{ij}\{z;x\},$$
$$S^1_{1j}\{z;x\}, \quad x^1 S^k_{11}\{z;x\}, \quad S^1_{11}\{z;x\} - \tfrac{n-1}{n+1}\tfrac{\alpha-1}{x^1}$$

are holomorphic for $2 \leq i, j, k \leq n$. (Formulae in [9, p.105] should be corrected as above. These can be proved by expressing $\partial z^i/\partial x^j$, $\partial^2 z^p/\partial x^i \partial x^j$ and $\partial x^i/\partial z^j$ in terms of v^j, u and α, and by substituting these expressions into the definition of the Schwarzian derivatives. Computation is lengthy but straightforward.)

(4) (relation with differential equations) Given an integrable system E_n of normal form, let $u_0, \dots, u_n$ be linearly independent solutions of E_n. Then the coefficients S^k_{ij} of E_n can be recovered from u as

$$S^k_{ij} = S^k_{ij}\{z;x\}, \quad \text{where } z = (u_1/u_0, \dots, u_n/u_0).$$

Conversely, given a non-degenerate map $x \mapsto z$, put $S^k_{ij} = S^k_{ij}\{z;x\}$ and define S^0_{ij} by the integrability condition above, and form a system E_n with coefficients S^k_{ij} and S^0_{ij}. Then the system is integrable and normal. For any $n+1$ linearly independent solutions $u_0, \dots, u_n$ of E_n, the ratios $u_1/u_0, \dots, u_n/u_0$ are projectively related to z.

3 Automorphisms of M

Let us define as in [6] six birational transformations $s_1, \dots, s_6$ in $x = (x^1, \dots, x^4)$:

$$\begin{aligned}
s_1 : (x^1, x^2, x^3, x^4) &\to \left(\frac{1}{x^1}, \frac{1}{x^2}, \frac{x^3}{x^1}, \frac{x^4}{x^2}\right), \\
s_2 : (x^1, x^2, x^3, x^4) &\to (x^3, x^4, x^1, x^2), \\
s_3 : (x^1, x^2, x^3, x^4) &\to \left(\frac{x^1 - x^3}{1 - x^3}, \frac{x^2 - x^4}{1 - x^4}, \frac{x^3}{x^3 - 1}, \frac{x^4}{x^4 - 1}\right), \\
s_4 : (x^1, x^2, x^3, x^4) &\to \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \frac{1}{x^3}, \frac{x^4}{x^3}\right), \\
s_5 : (x^1, x^2, x^3, x^4) &\to (x^2, x^1, x^4, x^3), \\
s_6 : (x^1, x^2, x^3, x^4) &\to \left(\frac{1}{x^1}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}\right).
\end{aligned}$$

If M is regarded as the configuration space of six points in $\mathbf{P}^2$, the transformation s_1, for example, corresponds to the interchange of the two points represented by the first two column vectors of the matrix x. Each s_i turns out to be a biregular involution on M, and they form a group G isomorphic to the Weyl group of type E_6; relation of the generators are given by the Coxeter graph

$$\begin{array}{ccccccccc}
s_1 & — & s_2 & — & s_3 & — & s_4 & — & s_5 \\
 & & & & | & & & & \\
 & & & & s_6 & & & &
\end{array}$$

If M is regarded as the moduli space of cubic surfaces, each transformation s_i takes cubic surfaces to the isomorphic ones but changes linearly their chosen cycles defining the period map f. Thanks to the projective invariance of the Schwarzian derivatives (see §2 (1)) our system E with coefficients

$$S^k_{ij}\{f;x\} =: S^k_{ij}(x) \in \mathbf{C}(x^1,\dots,x^4)$$

is invariant under the action of G. The invariance under $s \in G$ implies

$$S^k_{ij}(y) = S^k_{ij}\{x;y\} + \sum_{p,q,r=1}^{4} S^r_{pq}(x)\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r},$$

where $y = sx$, because the right-hand side equals $S^k_{ij}\{f;y\}$ (see §2 (2)), and the left hand side is the pull-back $s^*S^k_{ij}(x)$. The transformations $s = s_2, t = s_5$ and

$$u = ts : (x^1,x^2,x^3,x^4) \to (x^4,x^3,x^2,x^1)$$

give the identities:

$S^2_{11}(x) = S^1_{22}(tx)$	$S^3_{11}(x) = S^1_{33}(sx)$	$S^4_{11}(x) = S^1_{44}(ux)$
$S^2_{12}(x) = S^1_{12}(tx)$	$S^3_{12}(x) = S^1_{34}(sx)$	$S^4_{12}(x) = S^1_{34}(ux)$
$S^2_{13}(x) = S^1_{24}(tx)$	$S^3_{13}(x) = S^1_{13}(sx)$	$S^4_{13}(x) = S^1_{24}(ux)$
$S^2_{14}(x) = S^1_{23}(tx)$	$S^3_{14}(x) = S^1_{23}(sx)$	$S^4_{14}(x) = S^1_{14}(ux)$
$S^2_{22}(x) = S^1_{11}(tx)$	$S^3_{22}(x) = S^1_{44}(sx)$	$S^4_{22}(x) = S^1_{33}(ux)$
$S^2_{23}(x) = S^1_{14}(tx)$	$S^3_{23}(x) = S^1_{14}(sx)$	$S^4_{23}(x) = S^1_{23}(ux)$
$S^2_{24}(x) = S^1_{13}(tx)$	$S^3_{24}(x) = S^1_{24}(sx)$	$S^4_{24}(x) = S^1_{13}(ux)$
$S^2_{33}(x) = S^1_{44}(tx)$	$S^3_{33}(x) = S^1_{11}(sx)$	$S^4_{33}(x) = S^1_{22}(ux)$
$S^2_{34}(x) = S^1_{34}(tx)$	$S^3_{34}(x) = S^1_{12}(sx)$	$S^4_{34}(x) = S^1_{12}(ux)$
$S^2_{44}(x) = S^1_{33}(tx)$	$S^3_{44}(x) = S^1_{22}(sx)$	$S^4_{44}(x) = S^1_{11}(ux)$

4 Explicit form of the system E

The explicit form of the system is as follows:

The period map (developing map) $f : M \to B^4$ can be given by five solutions of the system E with the following coefficients $S^k_{ij} = S^k_{ij}(x^1,x^2,x^3,x^4)$. Since the coefficients S^k_{ij} $(k = 0,2,3,4)$ can be expressed by S^1_{ij} (see §§2 and 3), we list only S^1_{ij}.

$$\begin{aligned}
S^1_{23} &= \frac{x^1(x^1-1)P^1_{23}}{J}, \qquad S^1_{24} = \frac{x^1(x^1-1)P^1_{24}}{(x^2-x^4)J},\\
S^1_{34} &= S^1_{24}(x^1,x^3,x^2,x^4), \qquad S^1_{44} = \frac{x^1(x^1-1)P^1_{44}}{x^4(x^4-1)(x^3-x^4)(x^2-x^4)J},\\
S^1_{22} &= \frac{x^1(x^1-1)P^1_{22}}{x^2(x^2-1)(x^1-x^2)(x^2-x^4)J}, \qquad S^1_{33} = S^1_{22}(x^1,x^3,x^2,x^4),
\end{aligned}$$

where

$$\begin{aligned}
J &= (x^1x^4 - x^2x^3)\{(x^1-1)(x^4-1) - (x^2-1)(x^3-1)\}\\
&\quad \times\{x^1x^4(x^2-1)(x^3-1) - x^2x^3(x^1-1)(x^4-1)\},
\end{aligned}$$

$$
\begin{aligned}
P^1_{23} &= \frac{-1}{3}x^4(x^4-1)(x^1-x^3)(x^1-x^2),\\
P^1_{24} &= \frac{1}{3}(x^1-x^3)\{2x^4x^2x^3x^1 - x^3x^1x^2 + x^3(x^2)^2 - (x^4)^2x^3x^1 + (x^4)^2x^2x^3 - 2x^4x^3x^2\\
&\quad -(x^4)^2x^2x^1 + (x^4)^2x^1 - (x^2)^2(x^3)^2 + x^2(x^3)^2\},\\
P^1_{44} &= \frac{-1}{3}\{2x^4x^2x^3x^1 - x^3x^1x^2 + x^3(x^2)^2 - (x^4)^2x^3x^1 + (x^4)^2x^2x^3 - 2x^4x^3x^2\\
&\quad -(x^4)^2x^2x^1 + (x^4)^2x^1 - (x^2)^2(x^3)^2 + x^2(x^3)^2\}^2,\\
P^1_{22} &= \frac{1}{3}(x^4-x^2)J - \frac{1}{3}x^2(x^2-1)x^4(x^4-1)(x^1-x^2)(x^3-x^4)(x^1-x^3)^2,
\end{aligned}
$$

$$
\begin{aligned}
S^1_{11} &= \frac{-2}{5}\frac{1}{x^1} + \frac{-2}{5}\frac{1}{x^1-1} + \frac{-1}{15}\frac{1}{x^1-x^2} + \frac{-1}{15}\frac{1}{x^1-x^3} + R^1_{11},\\
S^1_{12} &= \frac{2}{15}\frac{1}{x^2} + \frac{2}{15}\frac{1}{x^2-1} + \frac{1}{5}\frac{1}{x^1-x^2} + \frac{2}{15}\frac{1}{x^2-x^4} + R^1_{12},\\
S^1_{13} &= S^1_{12}(x^1,x^3,x^2,x^4),\\
S^1_{14} &= \frac{2}{15}\frac{1}{x^4} + \frac{2}{15}\frac{1}{x^4-1} + \frac{-2}{15}\frac{1}{x^2-x^4} + \frac{-2}{15}\frac{1}{x^3-x^4} + R^1_{14},
\end{aligned}
$$

and (in the following, we again put $x^1=u, x^2=v, x^3=w, x^4=z$)

$$
\begin{aligned}
R^1_{11} &= -(-3wz^2u + 2wz^2v - wvzu - 3z^2uv + 4v^2w^2 + 2z^3u^2w + 2z^3vu^2\\
&\quad -zv^3w^2 - v^2w^3z + 2z^2u^2 + v^3w^3 + 6w^2uzv^2 - 6w^2uz^2v - 2wu^2z^2v\\
&\quad -6z^2v^2wu + 3v^2z^2w^2 + 2vz^3wu - 3w^2v^2u - 2wu^2z^2 + 2wvu^2z - z^3vw\\
&\quad +v^3wz - 3zv^2w^2 + 3z^2v^2u - 2zvw^2 + zvw^3 - 2zv^2w - 2z^2vu^2 - v^3w^2\\
&\quad -v^2w^3 + 11wvz^2u - 2z^3u^2 - 3vz^3u + 3uz^3 + 3uz^2w^2 - 3uz^3w)/15J,\\
R^1_{12} &= (4wz^2u - 3wvzu + v^2w^2 + 2z^3u^2w + v^2w^3z - 3z^2u^2 - 2z^3u^3\\
&\quad -w^2uzv^2 + 2w^2uz^2v - 4w^3uzv - 2wu^2z^2v + 2w^2u^2zv + w^3uv^2 - w^2u^2z^2\\
&\quad +2wu^3z^2 - 2u^3wz - u^2w^2v - w^2v^2u + 4wu^2z + w^2uv - 4wu^2z^2 + w^3uv\\
&\quad -zv^2w^2 - z^2vw^2 + 2z^2u^3 + zvw^2 + zvw^3 + z^2vu^2 - v^2w^3 + 3w^2uvz + v^2uwz\\
&\quad +2z^3u^2 - vw^3 - 4uzw^2 - 2uz^3w + 2zw^3u)/15J,\\
R^1_{14} &= (-7wvzu - 2v^2w^2 + 3v^3uw^2 - u^2vz + 4z^2u^2 - 3v^3w^3 + 2w^2uzv^2\\
&\quad -4wu^2z^2v - 6vzu^3w - 2w^2u^2zv - 2v^2zu^2w + 4vu^3z^2 + u^2v^2z - u^3vz + 3w^3uv^2\\
&\quad +4wu^3z^2 + 3u^3vw - u^3wz - 11w^2v^2u - 6u^2vw - wu^2z + 6w^2uv + 6v^2uw + u^2w^2v^2\\
&\quad -4wu^2z^2 + w^2u^2z - 3w^3uv - 3v^3uw + 17wvu^2z - zv^2w^2 - 4z^2u^3 - 4z^2vu^2\\
&\quad +3v^3w^2 + 3v^2w^3 + 4wvz^2u + u^3z)/15J.
\end{aligned}
$$

5 Procedure to get E

5.1 Local behavior of the system E

Applying (3) in §2 along $x^i=0$ and $x^i=1$, we get local behavior of our system E along these hyperplanes. By making coordinate changes, for example, $(x^1,\dots,x^4)\mapsto$

$(x^1 - x^2, x^2, \dots, x^4)$, we get local information of E also along $x^i = x^j$. We tabulate the local behavior of S^1_{ij}, since other coefficients can be expressed by these. The assertion (3) says more; for example,

$$S^1_{11} = -\frac{2}{5}\frac{1}{x^1} + (\text{holomorphic function along } x^1 = 0),$$

but we do not use such detailed properties. We just need the following tables, where

- h denotes a holomorphic function,
- $\circ$ denotes the term holomorphic relative to x^i, $x^i - 1$, $x^i - x^j$,
- $\bullet$ denotes the term of the form h/x^i, $h/(x^i - 1)$, $h/(x^i - x^j)$;

more precisely, S^1_{11} for example is holomorphic along the divisors $x^2 = 0$, $x^3 = 0$, $x^4 = 0$, $x^2 - 1 = 0$, $x^3 - 1 = 0$, $x^4 - 1 = 0$, $x^3 - x^4 = 0$, $x^2 - x^4 = 0$, and is singular along the divisors $x^1 = 0$, $x^1 - 1 = 0$, $x^1 - x^2 = 0$, and $x^1 - x^3 = 0$ so that it behaves like $h_1/x^1 + h_2/(x^1 - 1) + h_3/(x^1 - x^2) + h_4/(x^1 - x^3)$ where h_i are holomorphic along the respective divisors.

	x^1	x^2	x^3	x^4	$x^1 - 1$	$x^2 - 1$	$x^3 - 1$	$x^4 - 1$
S^1_{11}	$\bullet$	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$	$\circ$
S^1_{12}	$\circ$	$\bullet$	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$
S^1_{13}	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$
S^1_{14}	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$	$\circ$	$\bullet$
S^1_{22}	$x^1 \cdot h$	$\bullet$	$\circ$	$\circ$	$(x^1 - 1)h$	$\bullet$	$\circ$	$\circ$
S^1_{23}	$x^1 \cdot h$	$\circ$	$\circ$	$\circ$	$(x^1 - 1)h$	$\circ$	$\circ$	$\circ$
S^1_{24}	$x^1 \cdot h$	$\circ$	$\circ$	$\circ$	$(x^1 - 1)h$	$\circ$	$\circ$	$\circ$
S^1_{33}	$x^1 \cdot h$	$\circ$	$\bullet$	$\circ$	$(x^1 - 1)h$	$\circ$	$\bullet$	$\circ$
S^1_{34}	$x^1 \cdot h$	$\circ$	$\circ$	$\circ$	$(x^1 - 1)h$	$\circ$	$\circ$	$\circ$
S^1_{44}	$x^1 \cdot h$	$\circ$	$\circ$	$\bullet$	$(x^1 - 1)h$	$\circ$	$\circ$	$\bullet$

	$x^1 - x^2$	$x^3 - x^4$	$x^1 - x^3$	$x^2 - x^4$
S^1_{11}	$\bullet$	$\circ$	$\bullet$	$\circ$
S^1_{12}	$\bullet$	$\circ$	$\circ$	$\bullet$
S^1_{13}	$\circ$	$\bullet$	$\bullet$	$\circ$
S^1_{14}	$\circ$	$\bullet$	$\circ$	$\bullet$
S^1_{22}	$\bullet$	$\circ$	$\circ$	$\bullet$
S^1_{23}	$\circ$	$\circ$	$\circ$	$\circ$
S^1_{24}	$\circ$	$\circ$	$\circ$	$\bullet$
S^1_{33}	$\circ$	$\bullet$	$\bullet$	$\circ$
S^1_{34}	$\circ$	$\bullet$	$\circ$	$\circ$
S^1_{44}	$\circ$	$\bullet$	$\circ$	$\bullet$

We express each coefficient S^k_{ij} as a ratio of polynomials:

$$S^k_{ij} = \frac{\mathtt{P}[k,i,j]}{\mathtt{D}[k,i,j]}.$$

The local behavior above allows us to put

$$\begin{aligned}
\texttt{D}[1,1,1] &= x^1(x^1-1)(x^1-x^2)(x^1-x^3)J,\\
\texttt{D}[1,1,2] &= x^2(x^2-1)(x^2-x^4)(x^1-x^2)J,\\
\texttt{D}[1,1,3] &= x^3(x^3-1)(x^3-x^4)(x^1-x^3)J,\\
\texttt{D}[1,1,4] &= x^4(x^4-1)(x^3-x^4)(x^2-x^4)J,\\
\texttt{D}[1,2,2] &= x^2(x^2-1)(x^1-x^2)(x^2-x^4)J,\\
\texttt{D}[1,2,3] &= J,\\
\texttt{D}[1,2,4] &= (x^2-x^4)J,\\
\texttt{D}[1,3,3] &= x^3(x^3-1)(x^3-x^4)(x^1-x^3)J,\\
\texttt{D}[1,3,4] &= (x^3-x^4)J,\\
\texttt{D}[1,4,4] &= x^4(x^4-1)(x^3-x^4)(x^2-x^4)J,\\
\texttt{D}[1,i,j] &= \texttt{D}[1,j,i],
\end{aligned}$$

where J is defined in §4, and

$$\begin{aligned}
\texttt{P}[1,1,1] &= P^1_{11}, \quad \texttt{P}[1,1,2]=P^1_{12}, \quad \texttt{P}[1,1,3]=P^1_{13}, \quad \texttt{P}[1,1,4]=P^1_{14},\\
\texttt{P}[1,2,2] &= x^1(x^1-1)P^1_{22}, \quad \texttt{P}[1,2,3]=x^1(x^1-1)P^1_{23},\\
\texttt{P}[1,2,4] &= x^1(x^1-1)P^1_{24}, \quad \texttt{P}[1,3,3]=x^1(x^1-1)P^1_{33},\\
\texttt{P}[1,3,4] &= x^1(x^1-1)P^1_{34}, \quad \texttt{P}[1,4,4]=x^1(x^1-1)P^1_{44},\\
\texttt{P}[1,i,j] &= \texttt{P}[1,j,i].
\end{aligned}$$

where P^i_{jk} are polynomials. Note that since G acts on the fifteen divisors defined by

$$D = J(x^1-x^2)(x^1-x^3)(x^2-x4)(x^3-x^4)\prod_{j=1}^{4} x^j(x^j-1)$$

transitively, the denominators of the coefficients should have the factor J.

5.2 Relations coming out of the symmetry relative to G

As is explained in §3, our system is invariant under the action of G. For the elements s_2, and s_5, we got 2-term relations (see §3). In general, put

$$\texttt{R}[m,k,i,j] := S^k_{ij}(y) - S^k_{ij}\{x;y\} - \sum S^r_{pq}(x)\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial y^k}{\partial x^r},$$

that should be zero for all m, k, i, j, where $y = s_m(x)$, $m = 1,\dots,6$, denote the transforms. For example,

$$\begin{aligned}
\texttt{R}[1,1,1,1] = {} & 5(x^1)^7(x^2)^5\left(x^1-x^3\right)\left(x^1x^4-x^2x^3\right)P^1_{11}\left(\frac{1}{x^1},\frac{1}{x^2},\frac{x^3}{x^1},\frac{x^4}{x^2}\right)\\
& +4\left(x^1-1\right)\left(x^1-x^2\right)\left(x^3-1\right)\left(x^3-x^4\right)\left(-x^4+x^2+x^1x^4+x^3-x^1-x^2x^3\right)\\
& \times\left(x^1x^4+x^4x^3x^2-x^4x^1x^2-x^2x^3-x^4x^3x^1+x^3x^1x^2\right)\left(x^1-x^3\right)\left(x^1x^4-x^2x^3\right)\\
& +5\left(x^3-1\right)\left(x^3-x^4\right)P^1_{11}+10\left(x^1-1\right)\left(x^1-x^2\right)P^1_{13}\\
& +5x^3\left(x^1-1\right)^2\left(x^1-x^2\right)P^1_{33},
\end{aligned}$$

where $P_{ij}^k = P_{ij}^k(x^1, x^2, x^3, x^4)$. This gives a 4-term relation $\mathtt{R}[1,1,1,1] = 0$. A computation yields several 2-term, 3-term, and 4-term relations, the list of these is given in the appendix.

5.3 Bounds of degrees of the numerators

The 2-term relations and 3-term relations give bounds on degrees of P_{ij}^k. Write $P_{ij}^k = \sum a_{pqrs}(x^1)^p(x^2)^q(x^3)^r(x^4)^s$. Then the maximum possible values of p, q, r, and s derived from each transformation are as follows.

kij	s6				s1 and s3		s4	
	p	q	r	s	p+r	q+s	p+q	r+s
111	6	4	4	3	7	5	7	5
112	4	6	3	4	5	7	7	5
113	4	3	6	4	7	5	5	7
114	3	4	4	6	5	7	5	7
122	3	4	3	4	4	6	5	5
123	2	1	1	3	2	3	2	3
124	2	2	3	2	3	3	3	4
133	3	3	4	4	5	5	4	6
134	2	2	3	2	3	4	3	3
144	2	4	4	4	4	6	4	6

Let us see how to get the values above for the case $(k, i, j) = (1, 2, 3)$. First, noting that the first term of $\mathtt{R}[1,1,2,3]$ must be a polynomial, we get the bounds $p+r \leq 2$ and $q+s \leq 3$; the relation $\mathtt{R}[3,1,2,3] = 0$ gives the same bounds. Next, look at, say, $\mathtt{R}[4,1,2,3]$, which shows the bound $p+q \leq 2$ and $r+s \leq 3$. The relation $\mathtt{R}[6,1,2,3] = 0$ shows $p \leq 2$, $q \leq 1$, $r \leq 1$, and $s \leq 3$. For the remaining cases (k, i, j), it is enough to examine the expressions $\mathtt{R}[1,k,i,j]$, $\mathtt{R}[3,k,i,j]$, $\mathtt{R}[4,k,i,j]$, and $\mathtt{R}[6,k,i,j]$.

5.4 Solving the relations I

Among these relations, we first solve the 3-term relations to get expressions of P_{22}^1, P_{24}^1, etc. in terms of P_{12}^1, P_{11}^1, etc. In the following list, read " `kij` $\to$ `rpq [abcd] XX`" as P_{ij}^k can be written in terms of P_{pq}^r using the relation $\mathtt{R}[a,b,c,d] = 0$ and the expression obtained is named `XX`.

```
kij        rpq [abcd]      XX

124  ---> 113 [4224]       E1
     ---> 123 [4123]       E3

133  ---> 111 [1333]       A1

134  ---> 112 [1334]       B1
```

```
     ---> 114 [1114]     B3
     ---> 114 [3323]     B4
     ---> 123 [1123]     B5
     ---> 123 [3314]     B6

144  ---> 124 [1124]     C3
```

The expressions `XX` and the two new expressions `B3B4:=B3-B4` and `B5B6:=B5-B6` are listed below.

Notation. From now on, for notational simplicity, we write $x_1, x_2, \ldots$ in place of $x^1, x^2, \ldots$, (so x_1^3 stands for $(x_1)^3$) and as before P_{ij}^k for $P_{ij}^k(x_1, x_2, x_3, x_4)$.

$$
\begin{aligned}
\texttt{E1} &= -\frac{{x_3}^7{x_1}^5}{x_4x_2\,(x_4-1)\,(x_3-x_4)\,(x_1-1)}P_{13}^1(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3}) \\
&\quad +\frac{1}{x_4x_2\,(x_4-1)\,(x_3-x_4)\,(x_1-1)}P_{13}^1(x_2,x_1,x_4,x_3), \\
\texttt{E3} &= -\frac{x_3\,(x_2-x_4)}{x_4}P_{23}^1+\frac{{x_1}^2{x_3}^3\,(x_1x_4-x_2x_3)}{x_4}P_{23}^1(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3}), \\
\texttt{A1} &= -\frac{{x_1}^7{x_2}^5}{x_3\,(x_1-1)}P_{11}^1(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2})+\frac{1}{x_3\,(x_1-1)}P_{11}^1(x_3,x_4,x_1,x_2), \\
\texttt{B1} &= -\frac{{x_2}^7{x_1}^5}{x_4x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)}P_{12}^1(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2}) \\
&\quad +\frac{1}{x_4x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)}P_{12}^1(x_3,x_4,x_1,x_2), \\
\texttt{B3} &= -\frac{1}{x_4x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)}P_{14}^1 \\
&\quad +\frac{{x_2}^7{x_1}^5}{x_4x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)}P_{14}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2}), \\
\texttt{B4} &= -\frac{1}{x_1x_4\,(x_4-1)\,(x_3-1)\,(x_2-x_4)}P_{14}^1 \\
&\quad +\frac{(x_2-1)^7\,(x_1-1)^5}{x_1x_4\,(x_4-1)\,(x_3-1)\,(x_2-x_4)}P_{14}^1\left(\frac{x_1}{x_1-1},\frac{x_2}{x_2-1},\frac{x_1-x_3}{x_1-1},\frac{x_2-x_4}{x_2-1}\right), \\
\texttt{B5} &= -\frac{x_2\,(x_3-x_4)}{x_4}P_{23}^1+\frac{{x_1}^2{x_2}^3\,(x_1x_4-x_2x_3)}{x_4}P_{23}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2}), \\
\texttt{B6} &= -\frac{(x_3-x_4)\,(x_2-1)}{x_4-1}P_{23}^1 \\
&\quad -\frac{(x_2-1)^3\,(x_1-1)^2\,(-x_4+x_2+x_1x_4+x_3-x_1-x_2x_3)}{x_4-1} \\
&\quad\ P_{23}^1\left(\frac{x_1}{x_1-1},\frac{x_2}{x_2-1},\frac{x_1-x_3}{x_1-1},\frac{x_2-x_4}{x_2-1}\right), \\
\texttt{C3} &= -x_2\,(x_4-1)\,(x_3-x_4)\,P_{24}^1 \\
&\quad -{x_1}^3\,(x_2-x_4)\,{x_2}^3\,(x_1x_4-x_2x_3)\,P_{24}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2}), \\
\texttt{B3B4} &= (x_4-1)(x_2-x_4)(x_1-x_3)P_{14}^1
\end{aligned}
$$

$$+(x_4-1)(x_2-x_4)x_2^7x_1^6(x_3-1)P_{14}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})$$
$$-(x_4-1)(x_2-x_4)x_3(x_2-1)^7(x_1-1)^6P_{14}^1(\frac{x_1}{x_1-1},\frac{x_2}{x_2-1},\frac{x_1-x_3}{x_1-1},\frac{x_2-x_4}{x_2-1}),$$

$$\mathtt{B5B6} = (x_3-x_4)(x_2-x_4)P_{23}^1+x_1^2x_2^3(x_4-1)(x_1x_4-x_2x_3)P_{23}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})$$
$$+x_4(x_2-1)^3(x_1-1)^2(-x_4+x_2+x_1x_4+x_3-x_1-x_2x_3)$$
$$P_{23}^1(\frac{x_1}{x_1-1},\frac{x_2}{x_2-1},\frac{x_1-x_3}{x_1-1},\frac{x_2-x_4}{x_2-1}).$$

We need another expression of C3: insert the expression E3 into C3 and denote this expression by C3r:

$$\mathtt{C3r} = \frac{x_2x_3\,(x_4-1)\,(x_3-x_4)\,(x_2-x_4)}{x_4}P_{23}^1$$
$$-\frac{{x_2}^3{x_1}^2x_3\,(x_4-1)\,(x_2-x_4)\,(x_1x_4-x_2x_3)}{x_4}P_{23}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})$$
$$-\frac{x_2{x_1}^2{x_3}^3\,(x_4-1)\,(x_3-x_4)\,(x_1x_4-x_2x_3)}{x_4}P_{23}^1(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})$$
$$+\frac{{x_2}^3{x_3}^3\,(x_3-x_4)\,(x_2-x_4)\,(x_1x_4-x_2x_3)}{{x_1}^3x_4}P_{23}^1(x_1,\frac{x_1}{x_2},\frac{x_1}{x_3},\frac{x_1x_4}{x_2x_3}).$$

As a result, C3 and C3r give an expression of P_{44}^1 in terms of P_{23}^1.

We next insert expressions obtained above into the 2-term relations; then we get new relations without terms P_{24}^1, P_{33}^1, P_{34}^1, P_{44}^1. Among such relations, we need the following expression B6134 of R$[6,1,3,4]$, in which P_{34}^1 is replaced by B3:

$$\mathtt{B6134} = x_2\,x_1\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,{x_4}^6{x_3}^4P_{14}^1(x_1,x_2,\frac{x_1}{x_3},\frac{x_2}{x_4})$$
$$-{x_1}^3{x_2}^4\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P_{14}^1$$
$$+{x_1}^8{x_2}^{11}\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P_{14}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})$$
$$-{x_1}^6{x_3}^4{x_2}^8{x_4}^6\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P_{14}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4}).$$

5.5 Solving the relations II

Actual forms of P_{ij}^k can be found along the following procedures. The expression

$$\mathtt{R}[6,1,2,3] = {x_1}^2{x_4}^3x_2x_3P_{23}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+P_{23}^1$$

and the bounds of degrees show that P_{23}^1 has the form

$$b_{1001}(x_1x_4-x_1x_2x_3{x_4}^2)+b_{1002}(x_1{x_4}^2-x_1x_2x_3x_4)+b_{2001}({x_1}^2x_4-x_2x_3{x_4}^2)$$
$$+b_{2002}((x_1x_4)^2-x_2x_3x_4)+b_{1101}(x_1x_2x_4-x_1x_3{x_4}^2)+b_{1102}(x_1x_2{x_4}^2-x_1x_3x_4).$$

Insert this expression into the identity `B5B6`$=0$ and let all coefficients be zero: then, we get

$$P^1_{23} = c_1 \cdot x_4(x_4-1)(x_1-x_3)(x_1-x_2),$$

where c_1 is a constant. The expression `E3` gives

$$\begin{aligned} P^1_{24} &= c_1 \cdot \{-x_3(x_2-x_4)(x_4-1)(x_1-x_2)(x_1-x_3) \\ &\quad -(x_1x_4-x_2x_3)(x_3-x_4)(x_2-1)(x_1-x_3)\}. \end{aligned}$$

The expression `B5` gives

$$\begin{aligned} P^1_{34} &= c_1 \cdot \{-x_2(x_3-x_4)(x_4-1)(x_1-x_2)(x_1-x_3) \\ &\quad -(x_1x_4-x_2x_3)(x_2-x_4)(x_1-x_2)(x_3-1)\}. \end{aligned}$$

The expression `C3r` gives

$$\begin{aligned} P^1_{44} &= c_1 \cdot \Big(-x_2(x_4-1)(x_3-x_4)\{-x_3(x_2-x_4)(x_4-1)(x_1-x_2)(x_1-x_3) \\ &\quad -(x_1x_4-x_2x_3)(x_3-x_4)(x_2-1)(x_1-x_3)\} \\ &\quad -(1/{x_1}^3){x_2}^3(x_2-x_4)(x_1x_4-x_2x_3) \\ &\quad \times\{-x_3{x_1}^3(x_4-1)(x_2-x_4)(x_1-x_2)(x_3-1)/{x_2}^3 \\ &\quad -(x_3-x_4){x_1}^3(x_1x_4-x_2x_3)(x_2-1)(x_3-1)/{x_2}^3\}\Big). \end{aligned}$$

Then we can solve `R`$[4,2,3,4]=0$ to get P^1_{33} and `R`$[3,1,2,4]=0$ to get P^1_{22}. We see that $P^1_{33} = P^1_{22}(x_1,x_3,x_2,x_4)$.

Next, a look at symmetry shows that P^1_{14} has the form

$$\sum_{i=0}^{3}\sum_{j=0}^{4}\sum_{k=0}^{4}\sum_{m=0}^{6} a_{ijkm}{x_1}^i{x_2}^j{x_3}^k{x_4}^m,$$

where $i+k \le 5$, $j+m \le 7$, $i+j \le 5$, and $k+m \le 7$. Insert this expression into the equations

`B3B4`$=0$, `R`$[6,1,1,4]=0$, `R`$[4,1,1,4]=0$, `B6134`$=0$.

Then, we see that every coefficient can be written in terms of

$$c_1 := -7/5 - 4a_{3016} \quad \text{and} \quad c_2 := a_{3016}.$$

Now, from identities for $P^1_{23}, P^1_{24}, P^1_{34}, P^1_{44}, P^1_{22}, P^1_{33}, P^1_{14}$, we can determine $P^1_{12}, P^1_{13}, P^1_{11}$. The identities we need are

$$\begin{aligned} \texttt{S}[1,3,1,4] &= 0 \quad \text{for } P^1_{12}, \\ \texttt{S}[4,2,1,4] &= 0 \quad \text{for } P^1_{13}, \\ \texttt{S}[1,3,1,3] &= 0 \quad \text{for } P^1_{11}, \end{aligned}$$

where

$$\begin{aligned} \texttt{S}[1,3,1,4] &= \texttt{R}[1,3,1,4] \text{ with } P^1_{34} \text{ replaced by } \texttt{B1}, \\ \texttt{S}[4,2,1,4] &= \texttt{R}[4,2,1,4] \text{ with } P^1_{24} \text{ replaced by } \texttt{E1}, \\ \texttt{S}[1,3,1,3] &= \texttt{R}[1,3,1,3] \text{ with } P^1_{33} \text{ replaced by } \texttt{A1}. \end{aligned}$$

Explicit expressions are the following:

$$\texttt{S[1,3,1,4]} = -P^1_{14} + {x_1}^5{x_2}^7P^1_{12}(\frac{x_3}{x_1}, \frac{x_4}{x_2}, \frac{1}{x_1}, \frac{1}{x_2})$$
$$-{x_1}^2{x_2}^3x_4\ (x_4-1)\,(x_2-x_4)\,(x_1-x_3)\,(x_1\,x_4-x_2\,x_3)\,P^1_{23}(\frac{x_3}{x_1}, \frac{x_4}{x_2}, \frac{1}{x_1}, \frac{1}{x_2})$$
$$-x_1\,x_4\ (x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_2-x_4)\,P^1_{23}(x_3,x_4,x_1,x_2),$$
$$\texttt{S[4,2,1,4]} = -P^1_{14} + {x_1}^5{x_3}^7P^1_{13}(\frac{x_2}{x_1}, \frac{1}{x_1}, \frac{x_4}{x_3}, \frac{1}{x_3})$$
$$-{x_1}^2{x_3}^3x_4\ (x_4-1)\,(x_3-x_4)\,(x_1-x_2)\,(x_1\,x_4-x_2\,x_3)\,P^1_{23}(\frac{x_2}{x_1}, \frac{1}{x_1}, \frac{x_4}{x_3}, \frac{1}{x_3})$$
$$-x_1\,x_4\ (x_4-1)\,(x_3-x_4)\,(x_2-1)\,(x_2-x_4)\,P^1_{23}(x_2,x_1,x_4,x_3),$$
$$\texttt{S[1,3,1,3]} = \frac{2}{5}\ (x_1-1)\,(x_1-x_2)\,(x_3-1)\,(x_3-x_4)$$
$$\times\,(-x_4+x_2+x_1\,x_4+x_3-x_1-x_2\,x_3)$$
$$\times\,(x_1\,x_4+x_4\,x_3\,x_2-x_4\,x_1\,x_2-x_2\,x_3-x_4\,x_3\,x_1+x_3\,x_1\,x_2)\,(x_1-x_3)\,(x_1\,x_4-x_2\,x_3)$$
$$-\,(x_1-1)\,(x_1-x_2)\,P^1_{13}$$
$$+{x_1}^7{x_2}^5\,(x_1-1)\,(x_1-x_2)\,P^1_{11}(\frac{x_3}{x_1}, \frac{x_4}{x_2}, \frac{1}{x_1}, \frac{1}{x_2})$$
$$+{x_1}^7{x_2}^5\,(x_1-x_3)\,(x_1\,x_4-x_2\,x_3)\,P^1_{13}(\frac{x_3}{x_1}, \frac{x_4}{x_2}, \frac{1}{x_1}, \frac{1}{x_2})$$
$$+\,(x_3-1)\,(x_3-x_4)\,P^1_{13}(x_3,x_4,x_1,x_2).$$

In this way, we can determine P^k_{ij} up to the constant c_2. Insert these expressions into all the relations $\texttt{R}[m,k,i,j]=0$; then we find that

$$4+15c_2=0$$

is necessary and sufficient.

Remark. We required only part of the local behavior; especially, we did not give the exponents in advance. Under the invariance under G, the equation is *uniquely* determined, so are the exponents.

5.6 Integrability

Since we know *a priori* the existence of a system giving the developing map f, and that this system is invariant under G, the system E with the coefficients we found in this section must be the very equation we are looking for. So E must be integrable. However, the authors are afraid that the chain of logic is too long to let the readers believe. It is thus worthwhile to check the integrability condition given in §2 directly; indeed, the system is integrable.

6 Restriction of E along singular loci

It is known that the configuration space of six points in the projective line $\mathbf{P}^1$ can be uniformized by the 3-ball, in several ways, with the Appell-Lauricella hypergeometric system $E_D(a; b_1, b_2, b_3; c)$, defined below, as the uniformizing equation (see e.g. [8]). The most symmetric uniformization comes from the family of curves

$$t^3 = s(s-1)(s-y^1)(s-y^2)(s-y^3),$$

and the uniformizing equation is $E_D(2/3; 1/3, 1/3, 1/3; 4/3)$. Note that this family admits the action of the symmetry group of degree 6 which permutes the six points $\{0, 1, y^1, y^2, y^3, \infty\}$.

On the other hand, the hypersurface of $\mathbf{C}^4$, defined by the factor

$$x^1(x^2-1)(x^3-1)x^4 - (x^1-1)x^2x^3(x^4-1)$$

of $D(x)$, represents six points lying on a conic. Since any non-singular conic is isomorphic to $\mathbf{P}^1$, this locus identifies with the configuration space above. Recalling that every singular locus is equivalent under the action of the group G and that the isotropy subgroup of G with respect to any singular locus is isomorphic to the symmetric group of degree 6, we can naturally expect that the restriction of E along any singular locus is equivalent to $E_D(2/3; 1/3, 1/3, 1/3; 4/3)$.

The system $E_D(a; b_1, b_2, b_3; c)$ is defined to be

$$\begin{aligned} \frac{\partial^2 z}{\partial x^{i2}} &= -\left\{ \sum_{k\neq i} \frac{b_k x_k}{x^i(x^i-x^k)} + \frac{(a+b_i+1)x^i - c}{x^i(x^i-1)} \right\} \frac{\partial z}{\partial x^i} \\ &\quad + b_i \sum_{k\neq i} \frac{x^k(x^k-1)}{x^i(x^i-1)(x^i-x^k)} \frac{\partial z}{\partial x^k} - \frac{ab_i}{x^i(x^i-1)} \\ \frac{\partial^2 z}{\partial x^i \partial x^j} &= \frac{b_j}{x^i-x^j}\frac{\partial z}{\partial x^i} - \frac{b_i}{x^i-x^j}\frac{\partial z}{\partial x^j} \qquad \text{for} \quad i \neq j. \end{aligned}$$

This is the system satisfied by the Appell-Lauricella hypergeometric series defined by

$$\begin{aligned} &F_D(a; b_1, b_2, b_3; c|y^1, y^2, y^3) = \\ &\quad \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a, m_1+m_2+m_3)(b_1, m_1)(b_2, m_2)(b_3, m_3)}{(c, m_1+m_2+m_3)m_1!m_2!m_3!} (y^1)^{m_1}(y^2)^{m_2}(y^3)^{m_3}, \end{aligned}$$

where $(a, n) = a(a+1)\cdots(a+n-1)$. (For more details, see [7].)

Without loss of generality, we restrict our system E to the divisor $\{x^4 = 0\}$. We express solutions v of E as

$$v = (x^4)^\lambda (w(x^1, x^2, x^3) + w_1(x^1, x^2, x^3)x^4 + \cdots)$$

and find the exponent λ and the system of differential equations satisfied by w.

First, substituting the above expression into

$$\partial^2 v/(\partial x^4)^2 = S^1_{44}\partial v/\partial x^1 + S^2_{44}\partial v/\partial x^2 + S^3_{44}\partial v/\partial x^3 + S^4_{44}\partial v/\partial x^4 + S^0_{44}v,$$

and using the fact

$$(x^4 S^4_{44})|_{x^4=0} = -\frac{2}{5} \qquad ((x^4)^2 S^0_{44})|_{x^4=0} = -\frac{14}{225}$$

and that the orders of S^1_{44}, S^2_{44}, and S^3_{44} relative to x^4 are equal to -1, we get the equation

$$\lambda(\lambda-1) = -\frac{2}{5}\lambda - \frac{14}{225}.$$

This yields $\lambda = 2/15$ or $\lambda = 7/15$.

Next, substituting the above expression into

$$\partial^2 v/\partial x^1 \partial x^4 = S^1_{14}\partial v/\partial x^1 + S^2_{14}\partial v/\partial x^2 + S^3_{14}\partial v/\partial x^3 + S^4_{14}\partial v/\partial x^4 + S^0_{14}v,$$

and using the fact S^2_{14}, S^3_{14}, and S^4_{14} have no poles along $x^4 = 0$, we have

$$\lambda \partial w/\partial x^1 = (x^4 S^1_{14})|_{x^4=0}\partial w/\partial x^1 + \lambda S^4_{14}|_{x^4=0} w + (x^4 S^0_{14})|_{x^4=0} w.$$

Since we can see $(x^4 S^1_{14})|_{x^4=0} = 2/15$ and $(x^4 S^0_{14})|_{x^4=0} = -2/15 S^4_{14}|_{x^4=0}$, we conclude that $\lambda = 2/15$, which we now assume in the following. We get the same condition for the remaining cases for $\partial^2 v/\partial x^2 \partial x^4$ and $\partial^2 v/\partial x^3 \partial x^4$.

For each $1 \le i, j \le 3$, the equation has the form

$$(x^4)^\lambda \partial^2 w/\partial x^i \partial x^j = \left(\sum_{k=1}^3 S^k_{ij}\partial w/\partial x^k + \lambda S^4_{ij} w/x^4 + S^0_{ij} w\right)(x^4)^\lambda + (\text{higher terms relative to } x^4).$$

Since S^4_{ij} is divisible by x^4, we see that w satisfies the equation

$$\partial^2 w/\partial x^i \partial x^j = \sum_{k=1}^3 T^k_{ij}\partial w/\partial x^k + T^0_{ij} w,$$

where

$$T^k_{ij} = S^k_{ij}|_{x^4=0}, \qquad T^0_{ij} = \lambda(S^4_{ij}/x^4)|_{x^4=0} + S^0_{ij}|_{x^4=0}.$$

The denominators of the coefficients T^k_{ij} can be factored and the factors are seen to be x^1, x^2, x^3, x^1-1, x^2-1, x^3-1, x^1-x^2, x^1-x^3, and $x^1+x^2x^3-x^2-x^3$.

On the other hand, consider the six points $\{0, \infty, 1, y^1, y^2, y^3\}$ in the projective line, which we represent by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & y^1 & y^2 & y^3 \end{pmatrix}.$$

Through the Veronese embedding of the line into the projective plane, these six points can be seen as six points on a conic, which we represent by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & (y^1)^2 & (y^2)^2 & (y^3)^2 \\ 0 & 0 & 1 & y^1 & y^2 & y^3 \end{pmatrix}.$$

Under the left GL_3-action, this is equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 1-y^1 & 1-y^2 & 1-y^3 \\ 0 & 1 & 0 & (y^1)^2-y^1 & (y^2)^2-y^2 & (y^3)^2-y^3 \\ 0 & 0 & 1 & y^1 & y^2 & y^3 \end{pmatrix}.$$

Now we recall the fact (cf. [6]) that the involution

$$\# : (I_3, x_{ij}) \longrightarrow (I_3, \frac{1}{x_{ij}})$$

exchanges the divisor representing six points on a conic and the divisor representing six points that the 4th, 5th and 6th points are collinear. Since the 1st, 2nd and 6th points are collinear on the divisor $\{x^4 = 0\}$, we apply, to the matrix above, the involution $\#$ and then exchange the 1st (resp. 2nd) column and the 4th (resp. 5th) column, and finally, using again the left GL_3-action, we get

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & y^1/y^2 & (y^1 - y^3)/(y^2 - y^3) \\ 0 & 0 & 1 & 1 & 1/y^2 & 0 \end{pmatrix}.$$

So, we now introduce new coordinates (y^1, y^2, y^3) by

$$x^1 = \frac{y^1}{y^2}, \qquad x^2 = \frac{y^1 - y^3}{y^2 - y^3}, \qquad x^3 = 1/y^2; \quad \text{i.e.,}$$

$$y^1 = \frac{x^1}{x^3}, \qquad y^2 = \frac{1}{x^3}, \qquad y^3 = \frac{(x^1 - x^2)}{x^3(1 - x^2)}.$$

Furthermore, let u be the new unknown obtained by multiplying the factor

$$(y^1(y^1-1)y^2(y^2-1)y^3(y^3-1))^{-2/15}(y^2)^{3/5} \times (y^2 - y^3)^{4/15}(y^1 - y^3)^{-1/5}(y^1 - y^2)^{-1/3}$$

to the old unknown w. Then, we can see that the system relative to the unknown u and the coordinates y^i is identical with the system of Appell-Lauricella $E_D(2/3; 1/3, 1/3, 1/3; 4/3)$ in three variables.

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Appendix: List of relations $\mathtt{R}[m,k,i,j]$

The following is the list of the relations $\mathtt{R}[m,k,i,j] = 0$; for simplicity, we list the expressions of $\mathtt{R}[m,k,i,j]$. Due to consideration of symmetry, the expressions $\mathtt{R}[1,2,i,j]$, $\mathtt{R}[1,4,i,j]$, $\mathtt{R}[3,2,i,j]$, $\mathtt{R}[3,4,i,j]$, $\mathtt{R}[4,3,i,j]$, $\mathtt{R}[4,4,i,j]$, $\mathtt{R}[6,2,i,j]$, $\mathtt{R}[6,3,i,j]$, and $\mathtt{R}[6,4,i,j]$ are omitted. Always P_{ij}^k stands for $P_{ij}^k(x_1,x_2,x_3,x_4)$.

$$
\begin{aligned}
\mathtt{R}[1,1,1,1] = {} & 5{x_1}^7{x_2}^5\,(x_1-x_3)\,(x_1x_4-x_2x_3)\,P_{11}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2}) \\
& +4\,(x_1-1)\,(x_1-x_2)\,(x_3-1)\,(x_3-x_4)\,(-x_4+x_2+x_1x_4+x_3-x_1-x_2x_3) \\
& \quad \cdot(x_1x_4+x_4x_3x_2-x_4x_1x_2-x_2x_3-x_4x_3x_1+x_3x_1x_2)\,(x_1-x_3)\,(x_1x_4-x_2x_3) \\
& +5\,(x_3-1)\,(x_3-x_4)\,P_{11}^1+10\,(x_1-1)\,(x_1-x_2)\,P_{13}^1+5x_3\,(x_1-1)^2\,(x_1-x_2)\,P_{33}^1 \\
\mathtt{R}[1,1,1,2] = {} & 5{x_1}^5{x_2}^7\,(x_2-x_4)\,(x_1x_4-x_2x_3)\,P_{12}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2}) \\
& +-3\,(x_2-1)\,(x_4-1)\,(x_1-x_2)\,(x_3-x_4)\,(-x_4+x_2+x_1x_4+x_3-x_1-x_2x_3) \\
& \quad \cdot(x_1x_4+x_4x_3x_2-x_4x_1x_2-x_2x_3-x_4x_3x_1+x_3x_1x_2)\,(x_2-x_4)\,(x_1x_4-x_2x_3) \\
& +5\,(x_4-1)\,(x_3-x_4)\,P_{12}^1+5\,(x_2-1)\,(x_1-x_2)\,P_{14}^1 \\
& +5x_3x_2\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_1-1)\,(x_1-x_2)\,(x_3-x_4)\,P_{23}^1 \\
& +5x_4x_3\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_1-1)\,(x_1-x_2)\,P_{34}^1 \\
\mathtt{R}[1,1,1,3] = {} & {x_1}^7{x_2}^5P_{13}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})-P_{13}^1-x_3\,(x_1-1)\,P_{33}^1 \\
\mathtt{R}[1,1,1,4] = {} & {x_1}^5{x_2}^7P_{14}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})-P_{14}^1-x_4x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P_{34}^1 \\
\mathtt{R}[1,1,2,2] = {} & -{x_1}^4{x_2}^6\,(x_2-x_4)\,(x_1x_4-x_2x_3)\,P_{22}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2}) \\
& +x_2\,(x_4-1)\,(x_3-x_4)\,P_{22}^1+2x_2x_4\,(x_2-1)\,(x_1-x_2)\,(x_4-1)\,(x_3-x_4)\,P_{24}^1 \\
& +x_4\,(x_2-1)\,(x_1-x_2)\,P_{44}^1 \\
\mathtt{R}[1,1,2,3] = {} & {x_1}^2{x_2}^3\,(x_1x_4-x_2x_3)\,P_{23}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})-x_2\,(x_3-x_4)\,P_{23}^1-x_4P_{34}^1 \\
\mathtt{R}[1,1,2,4] = {} & -{x_1}^3{x_2}^3\,(x_2-x_4)\,(x_1x_4-x_2x_3)\,P_{24}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2}) \\
& -x_2\,(x_4-1)\,(x_3-x_4)\,P_{24}^1-P_{44}^1 \\
\mathtt{R}[1,1,3,3] = {} & -{x_1}^5{x_2}^5P_{33}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})+P_{33}^1 \\
\mathtt{R}[1,1,3,4] = {} & -{x_1}^3{x_2}^4P_{34}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})+P_{34}^1 \\
\mathtt{R}[1,1,4,4] = {} & -{x_1}^4{x_2}^6P_{44}^1(\frac{1}{x_1},\frac{1}{x_2},\frac{x_3}{x_1},\frac{x_4}{x_2})+P_{44}^1 \\
\mathtt{R}[1,3,1,1] = {} & -{x_1}^5{x_2}^5\,(x_1-x_3)^2\,(x_1x_4-x_2x_3)\,P_{33}^1(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2}) \\
& +(x_3-1)\,(x_3-x_4)\,P_{11}^1+2\,(x_1-x_2)\,(x_1-1)\,P_{13}^1+x_3\,(x_1-1)^2\,(x_1-x_2)\,P_{33}^1 \\
& -x_1\,(x_3-1)^2\,(x_3-x_4)\,P_{33}^1(x_3,x_4,x_1,x_2)-2\,(x_3-1)\,(x_3-x_4)\,P_{13}^1(x_3,x_4,x_1,x_2) \\
& -(x_1-x_2)\,(x_1-1)\,P_{11}^1(x_3,x_4,x_1,x_2)
\end{aligned}
$$

$$\mathtt{R}[1,3,1,2] = (x_4-1)\,(x_3-x_4)\,P^1_{12} + (x_2-1)\,(x_1-x_2)\,P^1_{14}(x_1,x_2,x_3,x_4)$$
$$+{x_1}^3{x_2}^4\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_1-x_3)\,(x_1x_4-x_2x_3)\,P^1_{34}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2})$$
$$+x_2x_3\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_1-1)\,(x_1-x_2)\,(x_3-x_4)\,P^1_{23}$$
$$+x_3x_4\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_1-1)\,(x_1-x_2)\,P^1_{34}$$
$$+x_1x_2\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_2-1)\,(x_2-x_4)\,P^1_{34}(x_3,x_4,x_1,x_2)$$
$$+x_1x_4\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_2-1)\,(x_2-x_4)\,(x_1-x_2)\,P^1_{23}(x_3,x_4,x_1,x_2)$$
$$-\,(x_4-1)\,(x_3-x_4)\,P^1_{14}(x_3,x_4,x_1,x_2) - (x_2-1)\,(x_1-x_2)\,P^1_{12}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[1,3,1,3] = 5{x_1}^7{x_2}^5\,(x_1-x_3)\,(x_1x_4-x_2x_3)\,P^1_{13}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2})$$
$$+2\,(x_1-1)\,(x_1-x_2)\,(x_3-1)\,(x_3-x_4)\,(-x_4+x_2+x_1x_4+x_3-x_1-x_2x_3)$$
$$\cdot\,(x_1x_4+x_4x_3x_2-x_4x_1x_2-x_2x_3-x_4x_3x_1+x_3x_1x_2)\,(x_1-x_3)\,(x_1x_4-x_2x_3)$$
$$-5\,(x_1-1)\,(x_1-x_2)\,P^1_{13} - 5x_3\,(x_1-1)^2\,(x_1-x_2)\,P^1_{33}$$
$$+5\,(x_3-1)\,(x_3-x_4)\,P^1_{13}(x_3,x_4,x_1,x_2) + 5\,(x_1-1)\,(x_1-x_2)\,P^1_{11}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[1,3,1,4] = -{x_1}^2{x_2}^3x_4\,(x_4-1)\,(x_2-x_4)\,(x_1-x_3)\,(x_1x_4-x_2x_3)\,P^1_{23}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2})$$
$$-P^1_{14} - x_4x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P^1_{34}$$
$$-x_1x_4\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_2-x_4)\,P^1_{23}(x_3,x_4,x_1,x_2) + P^1_{12}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[1,3,2,2] = -{x_1}^4{x_2}^6\,(x_2-x_4)\,(x_1-x_3)\,(x_1x_4-x_2x_3)\,P^1_{44}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2})$$
$$+x_2\,(x_4-1)\,(x_3-x_4)\,(x_1-1)\,P^1_{22} + x_4\,(x_2-1)\,(x_1-1)\,(x_1-x_2)\,P^1_{44}$$
$$+2x_2x_4\,(x_2-1)\,(x_1-1)\,(x_1-x_2)\,(x_4-1)\,(x_3-x_4)\,P^1_{24}$$
$$-x_2\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,P^1_{44}(x_3,x_4,x_1,x_2)$$
$$-2x_2x_4\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_2-1)\,(x_1-x_2)\,P^1_{24}(x_3,x_4,x_1,x_2)$$
$$-x_4\,(x_3-1)\,(x_2-1)\,(x_1-x_2)\,P^1_{22}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[1,3,2,3] = 5{x_1}^5{x_2}^7\,(x_2-x_4)\,(x_1x_4-x_2x_3)\,P^1_{14}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2})$$
$$-3\,(x_2-1)\,(x_4-1)\,(x_1-x_2)\,(x_3-x_4)\,(-x_4+x_2+x_1x_4+x_3-x_1-x_2x_3)$$
$$\cdot\,(x_1x_4+x_4x_3x_2-x_4x_1x_2-x_2x_3-x_4x_3x_1+x_3x_1x_2)\,(x_1x_4-x_2x_3)\,(x_2-x_4)$$
$$-5x_3x_2\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_1-1)\,(x_1-x_2)\,(x_3-x_4)\,P^1_{23}$$
$$-5x_4x_3\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_1-1)\,(x_1-x_2)\,P^1_{34}$$
$$+5\,(x_4-1)\,(x_3-x_4)\,P^1_{14}(x_3,x_4,x_1,x_2) + 5\,(x_2-1)\,(x_1-x_2)\,P^1_{12}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[1,3,2,4] = -{x_1}^3{x_2}^3\,(x_2-x_4)\,(x_1-x_3)\,(x_1x_4-x_2x_3)\,P^1_{24}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2})$$
$$-x_2\,(x_4-1)\,(x_3-x_4)\,(x_1-1)\,P^1_{24} + (1-x_1)P^1_{44}$$
$$+x_2\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,P^1_{24}(x_3,x_4,x_1,x_2) + (x_3-1)P^1_{22}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[1,3,3,3] = {x_1}^7{x_2}^5P^1_{11}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2}) + x_3\,(x_1-1)\,P^1_{33} - P^1_{11}$$
$$\mathtt{R}[1,3,3,4] = {x_2}^7{x_1}^5P^1_{12}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2}) + x_4x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P^1_{34} - P^1_{12}$$
$$\mathtt{R}[1,3,4,4] = -{x_1}^4{x_2}^6\,(x_1-x_3)\,P^1_{22}(\frac{x_3}{x_1},\frac{x_4}{x_2},\frac{1}{x_1},\frac{1}{x_2}) + (x_1-1)P^1_{44} + (1-x_3)P^1_{22}$$

$$\mathtt{R}[3,1,1,1] = -\left(x_4-1\right)^5\left(x_3-1\right)^7 P^1_{11}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$+P^1_{11} - x_3\left(-1+x_1\right)P^1_{33}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,1,1,2] = -\left(x_4-1\right)^7\left(x_3-1\right)^5 P^1_{12}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$+P^1_{12} + x_2x_3\left(-1+x_2\right)\left(x_2-x_4\right)\left(-1+x_1\right)P^1_{34}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,1,1,3] = -5\left(x_4-1\right)^5\left(x_3-1\right)^7\left(x_1-x_3\right)\left(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4\right)$$
$$\cdot P^1_{13}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$+2x_3\left(x_3-x_4\right)x_1\left(x_1x_4-x_2x_3\right)\left(x_1-x_2\right)$$
$$\cdot\left(-x_4x_1x_2-x_2x_3+x_1x_4-x_3x_4x_1+x_1x_2x_3+x_4x_2x_3\right)$$
$$\cdot\left(x_1-x_3\right)\left(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4\right)$$
$$+5x_3\left(x_3-x_4\right)P^1_{11}+5x_1\left(x_1-x_2\right)P^1_{13}$$
$$-5{x_3}^2\left(x_3-x_4\right)\left(-1+x_1\right)P^1_{33}(x_3,x_4,x_1,x_2)-5x_3\left(x_3-x_4\right)P^1_{13}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,1,1,4] = -5\left(x_4-1\right)^7\left(x_3-1\right)^5\left(x_2-x_4\right)\left(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4\right)$$
$$\cdot P^1_{14}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$-3x_4\left(x_3-x_4\right)x_2\left(x_1x_4-x_2x_3\right)\left(x_1-x_2\right)$$
$$\cdot\left(-x_4x_1x_2-x_2x_3+x_1x_4-x_3x_4x_1+x_1x_2x_3+x_4x_2x_3\right)$$
$$\cdot\left(x_2-x_4\right)\left(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4\right)$$
$$+5\left(x_3-x_4\right)x_4P^1_{12}+5x_2\left(x_1-x_2\right)P^1_{14}$$
$$+5x_3x_2x_4\left(-1+x_2\right)\left(x_2-x_4\right)\left(x_3-x_4\right)\left(-1+x_1\right)P^1_{34}(x_3,x_4,x_1,x_2)$$
$$+5x_3x_2x_4\left(x_4-1\right)\left(x_2-x_4\right)\left(-1+x_1\right)\left(x_1-x_2\right)\left(x_3-x_4\right)P^1_{23}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,1,2,2] = -\left(x_4-1\right)^6\left(x_3-1\right)^4\left(x_1-x_3\right)P^1_{22}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$+x_1P^1_{22}-x_3P^1_{44}$$
$$\mathtt{R}[3,1,2,3] = -x_2\left(x_4-1\right)^3\left(-1+x_2\right)\left(x_2-x_4\right)\left(x_3-1\right)^2\left(x_1-x_3\right)$$
$$\cdot\left(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4\right)P^1_{23}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$+P^1_{12}+x_1x_2\left(-1+x_2\right)\left(x_2-x_4\right)\left(x_1-x_2\right)\left(x_3-1\right)P^1_{23}$$
$$+x_2x_3\left(-1+x_2\right)\left(x_2-x_4\right)\left(-1+x_1\right)P^1_{34}(x_3,x_4,x_1,x_2)-P^1_{14}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,1,2,4] = \left(x_4-1\right)^3\left(x_3-1\right)^3\left(x_2-x_4\right)\left(x_1-x_3\right)\left(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4\right)$$
$$\cdot P^1_{24}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$+x_1P^1_{22}+x_1x_2\left(x_4-1\right)\left(x_1-x_2\right)P^1_{24}$$
$$-x_3P^1_{44}(x_3,x_4,x_1,x_2)-x_2x_3\left(x_4-1\right)\left(x_1-x_2\right)P^1_{24}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,1,3,3] = -\left(x_4-1\right)^5\left(x_3-1\right)^5\left(x_1-x_3\right)^2\left(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4\right)$$
$$\cdot P^1_{33}\left(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}\right)$$
$$+x_3\left(x_3-x_4\right)P^1_{11}+2x_1\left(x_1-x_2\right)P^1_{13}+{x_1}^2\left(x_3-1\right)\left(x_1-x_2\right)P^1_{33}$$
$$-{x_3}^2\left(-1+x_1\right)\left(x_3-x_4\right)P^1_{33}(x_3,x_4,x_1,x_2)$$
$$-2x_3\left(x_3-x_4\right)P^1_{13}(x_3,x_4,x_1,x_2)-x_1\left(x_1-x_2\right)P^1_{11}(x_3,x_4,x_1,x_2)$$

$$\begin{aligned}
\mathtt{R}[3,1,3,4] = {} & x_2x_4\,(x_4-1)^4\,(x_3-1)^3\,(x_2-x_4)\,(x_1-x_3)\,(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4) \\
& \quad \cdot P^1_{34}(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}) \\
& +(x_3-x_4)\,x_4P^1_{12}+x_2\,(x_1-x_2)\,P^1_{14} \\
& +x_4x_1x_2\,(-1+x_2)\,(x_2-x_4)\,(x_3-1)\,(x_3-x_4)\,(x_1-x_2)\,P^1_{23} \\
& +x_1x_2x_4\,(x_4-1)\,(x_2-x_4)\,(x_1-x_2)\,(x_3-1)\,P^1_{34} \\
& +x_3x_2x_4\,(-1+x_2)\,(x_2-x_4)\,(-1+x_1)\,(x_3-x_4)\,P^1_{34}(x_3,x_4,x_1,x_2) \\
& +x_3x_2x_4\,(x_4-1)\,(x_2-x_4)\,(-1+x_1)\,(x_1-x_2)\,(x_3-x_4)\,P^1_{23}(x_3,x_4,x_1,x_2) \\
& -(x_3-x_4)\,x_4P^1_{14}(x_3,x_4,x_1,x_2)-x_2\,(x_1-x_2)\,P^1_{12}(x_3,x_4,x_1,x_2) \\
\mathtt{R}[3,1,4,4] = {} & -(x_4-1)^6\,(x_3-1)^4\,(x_2-x_4)\,(x_1-x_3)\,(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4) \\
& \quad \cdot P^1_{44}(\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4},\frac{x_3}{x_3-1},\frac{x_4}{x_4-1}) \\
& +x_1x_4\,(-1+x_2)\,(x_3-x_4)\,P^1_{22}+x_2x_1\,(x_4-1)\,(x_1-x_2)\,P^1_{44} \\
& +2x_1x_2x_4\,(x_4-1)\,(-1+x_2)\,(x_1-x_2)\,(x_3-x_4)\,P^1_{24} \\
& -x_3x_4\,(-1+x_2)\,(x_3-x_4)\,P^1_{44}(x_3,x_4,x_1,x_2) \\
& -2x_3x_2x_4\,(x_4-1)\,(-1+x_2)\,(x_1-x_2)\,(x_3-x_4)\,P^1_{24}(x_3,x_4,x_1,x_2) \\
& -x_2x_3\,(x_4-1)\,(x_1-x_2)\,P^1_{22}(x_3,x_4,x_1,x_2) \\
\mathtt{R}[3,3,1,1] = {} & P^1_{33}(x_3,x_4,x_1,x_2) \\
& -(x_4-1)^5\,(x_3-1)^5\,P^1_{33}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4}) \\
\mathtt{R}[3,3,1,2] = {} & -P^1_{34}(x_3,x_4,x_1,x_2) \\
& -(x_4-1)^4\,(x_3-1)^3\,P^1_{34}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4}) \\
\mathtt{R}[3,3,1,3] = {} & x_3\,(-1+x_1)\,P^1_{33}(x_3,x_4,x_1,x_2)+P^1_{13}(x_3,x_4,x_1,x_2) \\
& -(x_4-1)^5\,(x_3-1)^7\,P^1_{13}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4}) \\
\mathtt{R}[3,3,1,4] = {} & (1-x_2)P^1_{34}(x_3,x_4,x_1,x_2)-(x_4-1)\,(x_1-x_2)\,P^1_{23}(x_3,x_4,x_1,x_2) \\
& +(x_4-1)^3\,(x_3-1)^2\,(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4) \\
& \qquad P^1_{23}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4}) \\
\mathtt{R}[3,3,2,2] = {} & -(x_4-1)^6\,(x_3-1)^4\,P^1_{44}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4}) \\
& \quad +P^1_{44}(x_3,x_4,x_1,x_2) \\
\mathtt{R}[3,3,2,3] = {} & -(x_4-1)^7\,(x_3-1)^5\,P^1_{14}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4}) \\
& -x_2x_3\,(-1+x_2)\,(x_2-x_4)\,(-1+x_1)\,P^1_{34}(x_3,x_4,x_1,x_2)+P^1_{14}(x_3,x_4,x_1,x_2) \\
\mathtt{R}[3,3,2,4] = {} & P^1_{44}(x_3,x_4,x_1,x_2)+x_2\,(x_4-1)\,(x_1-x_2)\,P^1_{24}(x_3,x_4,x_1,x_2) \\
& +(x_4-1)^3\,(x_3-1)^3\,(x_2-x_4)\,(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4) \\
& \qquad \cdot P^1_{24}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4})
\end{aligned}$$

$$\mathtt{R}[3,3,3,3] = -5\,(x_4-1)^5\,(x_3-1)^7\,(x_1-x_3)\,(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4)$$
$$\cdot P^1_{11}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4})$$
$$+4x_3\,(x_3-x_4)\,x_1\,(x_1x_4-x_2x_3)\,(x_1-x_2)$$
$$\cdot(-x_4x_1x_2-x_2x_3+x_1x_4-x_3x_4x_1+x_1x_2x_3+x_4x_2x_3)\,(x_1-x_3)$$
$$\cdot(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4)$$
$$+5{x_3}^2\,(x_3-x_4)\,(-1+x_1)\,P^1_{33}(x_3,x_4,x_1,x_2)$$
$$+10x_3\,(x_3-x_4)\,P^1_{13}(x_3,x_4,x_1,x_2)+5x_1\,(x_1-x_2)\,P^1_{11}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,3,3,4] = -5\,(x_4-1)^7\,(x_3-1)^5\,(x_2-x_4)\,(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4)$$
$$\cdot P^1_{12}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4})$$
$$-3x_4\,(x_3-x_4)\,x_2\,(x_1x_4-x_2x_3)\,(x_1-x_2)$$
$$\cdot(-x_4x_1x_2-x_2x_3+x_1x_4-x_3x_4x_1+x_1x_2x_3+x_4x_2x_3)$$
$$\cdot(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4)\,(x_2-x_4)$$
$$-5x_3x_2x_4\,(-1+x_2)\,(x_2-x_4)\,(x_3-x_4)\,(-1+x_1)\,P^1_{34}(x_3,x_4,x_1,x_2)$$
$$-5x_3x_2x_4\,(x_4-1)\,(x_2-x_4)\,(-1+x_1)\,(x_1-x_2)\,(x_3-x_4)\,P^1_{23}(x_3,x_4,x_1,x_2)$$
$$+5\,(x_3-x_4)\,x_4P^1_{14}(x_3,x_4,x_1,x_2)+5x_2\,(x_1-x_2)\,P^1_{12}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[3,3,4,4] = -\,(x_4-1)^6\,(x_3-1)^4\,(x_2-x_4)\,(x_1x_4-x_1+x_3-x_2x_3+x_2-x_4)$$
$$\cdot P^1_{22}(\frac{x_3}{x_3-1},\frac{x_4}{x_4-1},\frac{x_1-x_3}{1-x_3},\frac{x_2-x_4}{1-x_4})$$
$$+x_4\,(-1+x_2)\,(x_3-x_4)\,P^1_{44}(x_3,x_4,x_1,x_2)+x_2\,(x_4-1)\,(x_1-x_2)\,P^1_{22}(x_3,x_4,x_1,x_2)$$
$$+2x_2x_4\,(x_4-1)\,(-1+x_2)\,(x_3-x_4)\,(x_1-x_2)\,P^1_{24}(x_3,x_4,x_1,x_2)$$
$$\mathtt{R}[4,1,1,1] = 5{x_1}^7{x_3}^5\,(x_1-x_2)\,(x_1x_4-x_2x_3)\,P^1_{11}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})$$
$$+4\,(x_1-1)\,(x_2-1)\,(x_1-x_3)\,(x_2-x_4)\,(-x_4+x_3+x_1x_4+x_2-x_2x_3-x_1)$$
$$\cdot(x_4x_2x_3+x_1x_4-x_4x_1x_3-x_2x_3-x_4x_2x_1+x_2x_1x_3)\,(x_1-x_2)\,(x_1x_4-x_2x_3)$$
$$+5\,(x_2-1)\,(x_2-x_4)\,P^1_{11}+10\,(x_1-1)\,(x_1-x_3)\,P^1_{12}+5x_2\,(x_1-1)^2\,(x_1-x_3)\,P^1_{22}$$
$$\mathtt{R}[4,1,1,2] = {x_1}^7{x_3}^5P^1_{12}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})-P^1_{12}-x_2\,(x_1-1)\,P^1_{22}$$
$$\mathtt{R}[4,1,1,3] = 5{x_1}^5{x_3}^7\,(x_3-x_4)\,(x_1x_4-x_2x_3)\,P^1_{13}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})$$
$$-3\,(x_3-1)\,(x_4-1)\,(x_1-x_3)\,(x_2-x_4)\,(-x_4+x_3+x_1x_4+x_2-x_2x_3-x_1)$$
$$\cdot(x_4x_2x_3+x_1x_4-x_4x_1x_3-x_2x_3-x_4x_2x_1+x_2x_1x_3)\,(x_3-x_4)\,(x_1x_4-x_2x_3)$$
$$+5\,(x_4-1)\,(x_2-x_4)\,P^1_{13}+5\,(x_3-1)\,(x_1-x_3)\,P^1_{14}$$
$$+5x_2x_3\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_1-1)\,(x_1-x_3)\,(x_2-x_4)\,P^1_{23}$$
$$+5x_4x_2\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_1-1)\,(x_1-x_3)\,P^1_{24}$$
$$\mathtt{R}[4,1,1,4] = {x_1}^5{x_3}^7P^1_{14}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})-P^1_{14}-x_2x_4\,(x_4-1)\,(x_3-x_4)\,(x_1-1)\,P^1_{24}$$
$$\mathtt{R}[4,1,2,2] = -{x_1}^5{x_3}^5P^1_{22}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})+P^1_{22}$$
$$\mathtt{R}[4,1,2,3] = {x_1}^2{x_3}^3\,(x_1x_4-x_2x_3)\,P^1_{23}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})-x_3\,(x_2-x_4)\,P^1_{23}-x_4P^1_{24}$$
$$\mathtt{R}[4,1,2,4] = -{x_1}^3{x_3}^4P^1_{24}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})+P^1_{24}$$

$$\mathtt{R}[4,1,3,3] = -{x_1}^4{x_3}^6\,(x_3-x_4)\,(x_1x_4-x_2x_3)\,P^1_{33}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})$$
$$+x_3\,(x_4-1)\,(x_2-x_4)\,P^1_{33}+2x_3x_4\,(x_3-1)\,(x_1-x_3)\,(x_4-1)\,(x_2-x_4)\,P^1_{34}$$
$$+x_4\,(x_3-1)\,(x_1-x_3)\,P^1_{44}$$
$$\mathtt{R}[4,1,3,4] = -{x_1}^3{x_3}^3\,(x_3-x_4)\,(x_1x_4-x_2x_3)\,P^1_{34}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})$$
$$-x_3\,(x_4-1)\,(x_2-x_4)\,P^1_{34}-P^1_{44}$$
$$\mathtt{R}[4,1,4,4] = -{x_1}^4{x_3}^6P^1_{44}(\frac{1}{x_1},\frac{x_2}{x_1},\frac{1}{x_3},\frac{x_4}{x_3})+P^1_{44}$$
$$\mathtt{R}[4,2,1,1] = -{x_1}^5{x_3}^5\,(x_1-x_2)^2\,(x_1x_4-x_2x_3)\,P^1_{22}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})$$
$$+\,(x_2-1)\,(x_2-x_4)\,P^1_{11}+2\,(x_1-1)\,(x_1-x_3)\,P^1_{12}+x_2\,(x_1-1)^2\,(x_1-x_3)\,P^1_{22}$$
$$-x_1\,(x_2-1)^2\,(x_2-x_4)\,P^1_{22}(x_2,x_1,x_4,x_3)$$
$$-2\,(x_2-1)\,(x_2-x_4)\,P^1_{12}(x_2,x_1,x_4,x_3)-(x_1-1)\,(x_1-x_3)\,P^1_{11}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,1,2] = 5{x_1}^7{x_3}^5\,(x_1-x_2)\,(x_1x_4-x_2x_3)\,P^1_{12}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})$$
$$+2\,(x_1-1)\,(x_2-1)\,(x_1-x_3)\,(x_2-x_4)\,(-x_4+x_3+x_1x_4+x_2-x_2x_3-x_1)$$
$$\cdot\,(x_4x_2x_3+x_1x_4-x_4x_1x_3-x_2x_3-x_4x_2x_1+x_2x_1x_3)\,(x_1-x_2)\,(x_1x_4-x_2x_3)$$
$$-5\,(x_1-1)\,(x_1-x_3)\,P^1_{12}-5x_2\,(x_1-1)^2\,(x_1-x_3)\,P^1_{22}$$
$$+5\,(x_2-1)\,(x_2-x_4)\,P^1_{12}(x_2,x_1,x_4,x_3)+5\,(x_1-1)\,(x_1-x_3)\,P^1_{11}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,1,3] = {x_1}^3{x_3}^4\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_1-x_2)\,(x_1x_4-x_2x_3)\,P^1_{24}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})$$
$$+\,(x_4-1)\,(x_2-x_4)\,P^1_{13}+\,(x_3-1)\,(x_1-x_3)\,P^1_{14}$$
$$+x_2x_3\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_1-1)\,(x_1-x_3)\,(x_2-x_4)\,P^1_{23}$$
$$+x_2x_4\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_1-1)\,(x_1-x_3)\,P^1_{24}$$
$$+x_1x_3\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_2-1)\,(x_2-x_4)\,P^1_{24}(x_2,x_1,x_4,x_3)$$
$$+x_1x_4\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_2-1)\,(x_2-x_4)\,(x_1-x_3)\,P^1_{23}(x_2,x_1,x_4,x_3)$$
$$-\,(x_4-1)\,(x_2-x_4)\,P^1_{14}(x_2,x_1,x_4,x_3)-(x_3-1)\,(x_1-x_3)\,P^1_{13}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,1,4] = -{x_1}^2{x_3}^3x_4\,(x_4-1)\,(x_3-x_4)\,(x_1-x_2)\,(x_1x_4-x_2x_3)\,P^1_{23}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})$$
$$-P^1_{14}-x_2\,(x_4-1)\,x_4\,(x_3-x_4)\,(x_1-1)\,P^1_{24}$$
$$-x_1\,(x_4-1)\,x_4\,(x_2-1)\,(x_2-x_4)\,(x_3-x_4)\,P^1_{23}(x_2,x_1,x_4,x_3)+P^1_{13}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,2,2] = {x_1}^7{x_3}^5P^1_{11}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})+x_2\,(x_1-1)\,P^1_{22}-P^1_{11}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,2,3] = 5{x_1}^5{x_3}^7\,(x_3-x_4)\,(x_1x_4-x_2x_3)\,P^1_{14}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})$$
$$-3\,(x_3-1)\,(x_4-1)\,(x_1-x_3)\,(x_2-x_4)\,(-x_4+x_3+x_1x_4+x_2-x_2x_3-x_1)$$
$$\cdot\,(x_4x_2x_3+x_1x_4-x_4x_1x_3-x_2x_3-x_4x_2x_1+x_2x_1x_3)\,(x_1x_4-x_2x_3)\,(x_3-x_4)$$
$$-5x_2x_3\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_1-1)\,(x_1-x_3)\,(x_2-x_4)\,P^1_{23}$$
$$-5x_4x_2\,(x_4-1)\,(x_3-1)\,(x_3-x_4)\,(x_1-1)\,(x_1-x_3)\,P^1_{24}$$
$$+5\,(x_4-1)\,(x_2-x_4)\,P^1_{14}(x_2,x_1,x_4,x_3)+5\,(x_3-1)\,(x_1-x_3)\,P^1_{13}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,2,4] = {x_1}^5{x_3}^7P^1_{13}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})+x_2x_4\,(x_4-1)\,(x_3-x_4)\,(x_1-1)\,P^1_{24}$$
$$-P^1_{13}(x_2,x_1,x_4,x_3)$$

$$\mathtt{R}[4,2,3,3] = -{x_1}^4{x_3}^6\,(x_3-x_4)\,(x_1-x_2)\,(x_1x_4-x_2x_3)\,P^1_{44}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})$$
$$+x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P^1_{33}$$
$$+2x_4x_3\,(x_3-1)\,(x_1-1)\,(x_1-x_3)\,(x_4-1)\,(x_2-x_4)\,P^1_{34}$$
$$+x_4\,(x_3-1)\,(x_1-1)\,(x_1-x_3)\,P^1_{44}$$
$$-x_3\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,P^1_{44}(x_2,x_1,x_4,x_3)$$
$$-2x_3x_4\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,(x_3-1)\,(x_1-x_3)\,P^1_{34}(x_2,x_1,x_4,x_3)$$
$$-x_4\,(x_3-1)\,(x_2-1)\,(x_1-x_3)\,P^1_{33}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,3,4] = -{x_1}^3{x_3}^3\,(x_3-x_4)\,(x_1-x_2)\,(x_1x_4-x_2x_3)\,P^1_{34}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})$$
$$-x_3\,(x_4-1)\,(x_2-x_4)\,(x_1-1)\,P^1_{34}+(1-x_1)P^1_{44}$$
$$+x_3\,(x_4-1)\,(x_2-1)\,(x_2-x_4)\,P^1_{34}(x_2,x_1,x_4,x_3)+(x_2-1)P^1_{33}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[4,2,4,4] = -{x_1}^4{x_3}^6\,(x_1-x_2)\,P^1_{33}(\frac{x_2}{x_1},\frac{1}{x_1},\frac{x_4}{x_3},\frac{1}{x_3})+(x_1-1)P^1_{44}$$
$$+(1-x_2)P^1_{33}(x_2,x_1,x_4,x_3)$$
$$\mathtt{R}[6,1,1,1] = 5{x_1}^6{x_2}^4{x_3}^4{x_4}^3P^1_{11}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+5P^1_{11}$$
$$+6\,(x_1-1)\,(x_1-x_2)\,(x_1-x_3)\,(x_1x_4-x_2x_3)$$
$$\cdot(-x_2x_3+x_4x_2x_3+x_1x_2x_3+x_1x_4-x_1x_4x_3-x_1x_4x_2)\,(x_3+x_2-x_2x_3-x_4-x_1+x_1x_4)$$
$$\mathtt{R}[6,1,1,2] = 5{x_2}^6{x_4}^4{x_1}^4{x_3}^3P^1_{12}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+5P^1_{12}$$
$$-2\,(x_2-1)\,(x_2-x_4)\,(x_1-x_2)\,(x_1x_4-x_2x_3)$$
$$\cdot(-x_2x_3+x_4x_2x_3+x_1x_2x_3+x_1x_4-x_1x_4x_3-x_1x_4x_2)\,(x_3+x_2-x_2x_3-x_4-x_1+x_1x_4)$$
$$\mathtt{R}[6,1,1,3] = 5{x_3}^6{x_4}^4{x_1}^4{x_2}^3P^1_{13}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+5P^1_{13}(x_1,x_2,x_3,x_4)$$
$$-2\,(x_3-1)\,(x_3-x_4)\,(x_1-x_3)\,(x_1x_4-x_2x_3)$$
$$\cdot(-x_2x_3+x_4x_2x_3+x_1x_2x_3+x_1x_4-x_1x_4x_3-x_1x_4x_2)\,(x_3+x_2-x_2x_3-x_4-x_1+x_1x_4)$$
$$\mathtt{R}[6,1,1,4] = 5{x_4}^6{x_3}^4{x_2}^4{x_1}^3P^1_{14}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+5P^1_{14}$$
$$-2\,(x_4-1)\,(x_3-x_4)\,(x_2-x_4)\,(x_1x_4-x_2x_3)$$
$$\cdot(-x_2x_3+x_4x_2x_3+x_1x_2x_3+x_1x_4-x_1x_4x_3-x_1x_4x_2)\,(x_3+x_2-x_2x_3-x_4-x_1+x_1x_4)$$
$$\mathtt{R}[6,1,2,2] = -{x_1}^3{x_2}^4{x_4}^4{x_3}^3P^1_{22}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+P^1_{22}$$
$$\mathtt{R}[6,1,2,3] = {x_1}^2{x_4}^3x_2x_3P^1_{23}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+P^1_{23}$$
$$\mathtt{R}[6,1,2,4] = -{x_1}^2{x_3}^3{x_2}^2{x_4}^2P^1_{24}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+P^1_{24}$$
$$\mathtt{R}[6,1,3,3] = -{x_3}^4{x_4}^4{x_1}^3{x_2}^3P^1_{33}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+P^1_{33}$$
$$\mathtt{R}[6,1,3,4] = -{x_2}^3{x_1}^2{x_3}^2{x_4}^2P^1_{34}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+P^1_{34}$$
$$\mathtt{R}[6,1,4,4] = -{x_1}^2{x_4}^4{x_3}^4{x_2}^4P^1_{44}(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4})+P^1_{44}$$