Modified $\mathcal{A}$-hypergeometric Systems

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Abstract We introduce a modification of $\mathcal{A}$-hypergeometric systems (GKZ systems) by applying a change of variables for Gröbner deformations and study its Gröbner basis and indicial polynomial along the exceptional hypersurface.
Key words: $\mathcal{A}$-hypergeometric system, GKZ system, indicial polynomial, Gröbner basis, modifed $\mathcal{A}$-hypergeometric system
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1 Introduction

An $\mathcal{A}$-hypergeometric system is a system of multidimensional hypergeometric partial differential equations defined by a matrix A and a set of parameters β. It is defined on $(y_1, \ldots, y_n)$ space. We will denote such a system by $H_A(\beta)$. By specializing parameters and some independent variables, we obtain classical hypergeometric differential equations in one or several variables. Ever since the study of Gel'fand, Zelevinsky, and Kapranov [3], studies of $\mathcal{A}$-hypergeometric systems (GKZ systems) have attracted the interest of many mathematicians, who wish to understand hypergeometric differential equations in a general way. We refer the reader to the book [11] for a review of important studies on $\mathcal{A}$-hypergeometric systems up to 2000, and the recent papers [5] and [12] and their references for recent advances. We also note that these studies have interacted fruitfully with the frontiers of computational commutative algebra and computational D-modules (where D denotes the ring of differential operators with polynomial coefficients).

In this short paper, we introduce a modified version of $\mathcal{A}$-hypergeometric systems and indicate a first step for studying them. The original system is defined on $y = (y_1, \ldots, y_n)$ space, whereas the modified system is defined on $(t, x_1, \ldots, x_n)$ space with an additional variable t. Let us sketch our idea for introducing the modified system. We consider the direct sum of the $\mathcal{A}$-hypergeometric D-module on y space and the D-module $D/D \cdot s\partial_s$ on s-space. For the weight vector $w \in \mathbf{Z}^n$, the original system restricted on $\mathbf{C}^n \times \mathbf{C}^*$ is transformed into the modified system on (t, x) space by the map

$$\mathbf{C}^n \times \mathbf{C}^* \ni (y_1, \ldots, y_n, s) \mapsto (x_1, \ldots, x_n, t) \in \mathbf{C}^n \times \mathbf{C}^*$$

where

$$(x_1, \ldots, x_n, t) = (s^{-w_1} y_1, \ldots, s^{-w_n} y_n, s)$$

and

$$(y_1, \ldots, y_n, s) = (t^{w_1} x_1, \ldots, t^{w_n} x_n, t)$$

(see [9] and [11] on this transformation). The transformed system can be naturally extended on $\mathbf{C}^{n+1}$. The system is firstly defined on $t \neq 0$ and is extended to the whole space $\mathbf{C}^{n+1}$ including $t = 0$. Intuitively speaking, the hypersurface $t = 0$ is analogous to the exceptional hypersurface of a blowing-up operation. We rigorously define the modified system in the next section.

We are interested in how solutions behave near the exceptional hypersurface $t = 0$. We will study the indicial polynomial along $t = 0$ as a first step to performing local and global analyses of the modified system.

2 Definition and Holonomic Rank of Modified $\mathcal{A}$-hypergeometric systems

Let $A = (a_{ij})$ be a $d \times n$-matrix whose elements are integers and $w = (w_1, \ldots, w_n)$ a vector of integers. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$. Set

$$\tilde{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ & \cdots & & 0 \\ a_{dn} & \cdots & a_{dn} & 0 \\ w_1 & \cdots & w_n & 1 \end{pmatrix}.$$

Definition 1 We call the following system of differential equations $H_{A,w}(\beta)$ a *modified $\mathcal{A}$-hypergeometric system*:

$$\begin{aligned} \left(\sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \right) \bullet f &= 0, \qquad (i = 1, \ldots, d) \\ \left(\sum_{j=1}^{n} w_j x_j \partial_j - t\partial_t \right) \bullet f &= 0, \\ \left(\prod_{i=1}^{n} \partial_i^{u_i} t^{u_{n+1}} - \prod_{j=1}^{n} \partial_j^{v_j} t^{v_{n+1}} \right) \bullet f &= 0. \qquad (u, v \in \mathbf{N}_0^{n+1} \text{ run over all } u, v \text{ such that } \tilde{A}u = \tilde{A}v) \end{aligned}$$

Here, $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbf{C}^d$ are parameters.

Let $I_{\tilde{A}}$ be the toric ideal generated by

$$\prod_{i=1}^{n} \partial_i^{u_i} t^{u_{n+1}} - \prod_{j=1}^{n} \partial_j^{v_j} t^{v_{n+1}} \qquad (u, v \in \mathbf{N}_0^{n+1} \text{ run over all } u, v \text{ such that } \tilde{A}u = \tilde{A}v) \tag{1}$$

in $\mathbf{C}[\partial_1, \ldots, \partial_n, t]$. Since $\mathbf{C}[\partial_1, \ldots, \partial_n, t]/I_{\tilde{A}}$ is an integral domain and t^m does not belong to the toric ideal, we have

$$I_{\tilde{A}} = I_{\tilde{A}}^{sat} := (I_{\tilde{A}} : t^\infty) := \{\ell \mid t^m \ell \in I_{\tilde{A}} \text{ for a non-negative integer } m\} \tag{2}$$

This fact will be used in the proof of Theorem 2.

We note that the matrix $\tilde{A}$ with $w = (1, \ldots, 1)$ was introduced in [7] to construct a fundamental set of convergent series solutions.

Throughout this paper, we will use facts shown in [11]. However, they are limited to only a small part of this book and this paper is written to be understandable with an elementary knowledge on Gröbner bases in the ring of differential operators and basics on D-modules.

Before starting an algebraic discussion about the modified $\mathcal{A}$-hypergeometric system, we informally discuss a bit about an integral representation of the modified system to explain about a relation between modified systems and classical hypergeometric functions. Let a_i be the i-th column vector of the matrix A and $F(\beta, x, t)$ the integral

$$F(\beta, x, t) = \int_C \exp\left(\sum_{i=1}^n x_i t^{w_i} s^{a_i}\right) s^{-\beta-1} ds, \qquad s = (s_1, \ldots, s_d),\ \beta = (\beta_1, \ldots, \beta_d).$$

Here, $s^{a_i} = \prod_{j=1}^d s_j^{a_{ji}}$, $s^{-\beta-1} = \prod_{j=1}^d s_j^{-\beta_j - 1}$, $ds = ds_1 \cdots ds_d$. The integral $F(\beta, x, t)$ satisfies the modified $\mathcal{A}$-hypergeometric system "formally", which can be shown by integration by parts (see, e.g., [11, 221–222]). We use the word "formally", because we have no general and rigorous description about the cycle C except the case of $d = 1$ [4]. However, the integral representation gives an intuitive description of what are solutions of modified $\mathcal{A}$-hypergeometric systems. We note that if $a_{di} = 1$ for all i, we also have the following "formal" integral representation

$$\begin{aligned} F(\beta, x, t) &= \int_C \left(\sum_{i=1}^n x_i t^{w_i} \tilde{s}^{\tilde{a}_i}\right)^{-\beta_d} \tilde{s}^{-\tilde{\beta}-1} d\tilde{s}, \\ &\quad \tilde{a}_i = (a_{1i}, \ldots, a_{d-1,i})^T, \tilde{s} = (s_1, \ldots, s_{d-1}),\ \tilde{\beta} = (\beta_1, \ldots, \beta_{d-1}). \end{aligned}$$

We denote by D the ring of differential operators $\mathbf{C}\langle x_1, \ldots, x_n, t, \partial_1, \ldots, \partial_n, \partial_t\rangle$. We will regard the modified $\mathcal{A}$-hypergeometric system as the left ideal in D. We will denote by $H_{A,w}(\beta)$ this left ideal as long as no confusion arises and call it the *modified $\mathcal{A}$-hypergeometric ideal*.

A set of generators of $H_{A,w}(\beta)$ can be computed by computing a set of generators of the toric ideal $I_{\tilde{A}}$. An algorithm of computing a set of generators of the toric ideal is given in the book [10, Algorithm 4.5].

Here is a log of a session with the computer algebra system `kan/sm1` [8] to obtain a set of generators of $H_{A,w}(\beta)$ for $A = (-1, 1, 2)$, $w = (-2, -1, 0)$ and $\beta = (0)$.

```
(cohom.sm1) run
[ [[-1,1,2]], [-2,-1,0], [0]] mgkz ::
 [-x1*Dx1+x2*Dx2+2*x3*Dx3, -2*x1*Dx1-x2*Dx2-x4*Dx4,
  x4^2*Dx2^2-Dx3,-x4^3*Dx1*Dx2+1,x4*Dx1*Dx3-Dx2,-x4*Dx2^3+Dx1*Dx3^2,-Dx1^2*Dx3^3+Dx2^4]
```

The command `mgkz` outputs a set of generators of $H_{A,w}(\beta)$. Here, `x1`, `x2`, `x3`, `x4` stand for x_1, x_2, x_3 and t respectively and `Dxi` stands for ∂_i.

The rank of $H_{A,w}(\beta)$ is the dimension of the $\mathbf{C}$-vector space of the classical solutions of $H_{A,w}(\beta)$ and is denoted by $\mathrm{rank}\,(H_{A,w}(\beta))$ [11, p.31].

Theorem 1 *1. The left D-module $D/H_{A,w}(\beta)$ is holonomic.*

2. The rank of $H_{A,w}(\beta)$ agrees with the rank of $H_A(\beta)$ for any w.

Proof. (1) We apply the *Laplace transformation* with respect to the variable t ($t \mapsto -\partial_{t'}$, $\partial_t \mapsto t'$) for the modified $\mathcal{A}$-hypergeometric ideal $H_{A,w}(\beta)$. Then, the transformed system is nothing but $\mathcal{A}$-hypergeometric ideal for the matrix $\tilde{A}$ and the parameter vector $(\beta_1, \ldots, \beta_n, -1)$. It is known that the transformed system is holonomic, then the original system is also holonomic by showing the Hilbert polynomials with respect to the Bernstein filtration of each system agree.

(2) We consider the biholomorphic map φ on $\mathbf{C}^n \times \mathbf{C}^*$

$$\mathbf{C}^n \times \mathbf{C}^* \ni (y_1, \ldots, y_n, s) \mapsto (s^{-w_1} y_1, \ldots, s^{-w_n} y_n, s) = (x_1, \ldots, x_n, t) \in \mathbf{C}^n \times \mathbf{C}^* \quad (3)$$

The map φ induces a correspondence of differential operators on $\mathbf{C}^n \times \mathbf{C}^*$

$$\begin{aligned} \frac{\partial}{\partial y_i} &= t^{-w_i} \frac{\partial}{\partial x_i} \\ -s\frac{\partial}{\partial s} &= -t\frac{\partial}{\partial t} + \sum_{j=1}^{n} w_n x_n \frac{\partial}{\partial x_i} \end{aligned}$$

Consider a left ideal H_Y in $D_Y = \mathbf{C}\langle y_1, \ldots, y_n, s, \partial_{y_1}, \ldots, \partial_{y_n}, \partial_s \rangle$ generated by $H_A(\beta)$ and $s\partial_s$. The holonomic rank of D_Y/H_Y is that of $H_A(\beta)$. We can see that the image of $\mathcal{D}_Y/\mathcal{D}_Y H_Y$ by the biholomorphic map φ on $\mathbf{C}^n \times \mathbf{C}^*$ is $\mathcal{D}_X/\mathcal{D}_X H_{A,w}(\beta)$ by utilizing the correspondence of differential operators. Here, $\mathcal{D}_X$ and $\mathcal{D}_Y$ denote the sheaves of differential operators on $\mathbf{C}^n \times \mathbf{C}^*$ of (y, s)-space and (x, t)-space respectively. Since the holonomic rank agrees with the multiplicity of the zero section of the characteristic variety at generic points, the holonomic ranks of the both systems agree [11, pp 28–40]. Q.E.D.

We denote by $\mathrm{vol}(A)$ the normalized volume of the convex hull of the column vectors of matrix A and the origin [11, p.50].

Corollary 1 $\mathrm{rank}\,(H_A(\beta)) \geq \mathrm{vol}(A)$

Proof. When A has $(1, 1, \ldots, 1)$ in its row space (A is homogeneous), $\mathrm{rank}\,(H_A(\beta)) \geq \mathrm{vol}(A)$ holds [11, Theorem 3.5.1], which is proved by utilizing that $H_A(\beta)$ is regular holonomic and by constructing $\mathrm{vol}(A)$ many series solutions. Put $w = (1, 1, \ldots, 1)$ in the modified system $H_{A,w}(\beta)$. Then, we have $\mathrm{rank}\,(H_{A,w}(\beta)) \geq \mathrm{vol}(A)$. Hence, Theorem 1 gives the conclusion. Q.E.D.

Note that the upper semi continuity theorem of holonomic rank of [5] also gives this result.

The aim of this paper is to determine the indicial polynomial of $H_{A,w}(\beta)$ along $t = 0$ and construct formal series solutions standing for roots of the indicial polynomial. Before starting a systematic discussion, we show one example of construction of a formal series solution.

Example 1 We take $A = (1, 3)$, $\beta = (-1)$, and $w = (-1, 0)$ (*Airy type integral*) [11, p.223]. The monomial $x_1^{-3m-1} x_2^m t^{3m+1}$ satisfies the two first order equations of the Definition 1. Then, we can expect a superposition of monomials of this form is a formal solution. The ideal $I_{\tilde{A}}$ is generated by $\partial_2 - t^3\partial_1^3$. Put $f = \sum_{m=0}^\infty d_m x_1^{-3m-1} x_2^m t^{3m+1}$ and determine d_m by utilizing $\partial_2 - t^3\partial_1^3$. Then, we can see that d_m satisfies the following recurrence

$$d_0 = 1, d_{m+1} = \frac{-(3m+1)(3m+2)(3m+3)}{m} d_m$$

The divergent series

$$\begin{aligned} f(x;t) &= \sum_{m=0}^{\infty} \left(d_m x_1^{-3m-1} x_2^m \right) t^{3m+1} \\ &= \sum_{m=0}^{\infty} \left((-1)^m \frac{\Gamma(3m+1)}{\Gamma(m+1)} x_1^{-3m-1} x_2^m \right) t^{3m+1} \end{aligned} \qquad (4)$$

is a formal solution of the modified system. Fix a point $(x_1, x_2) = (a_1, a_2)$ such that $a_1, a_2 \neq 0$. Then this is a Gevrey formal power series solution at $(a_1, a_2, 0)$ along $t = 0$ in the class $s = 1 + 2/3$ from the definition of Gevrey series.

The slope of a holonomic system is a set of numbers each of which stands for discontinuity of a one parameter family of micro-characteristic varieties [1]. The slope of this system can be computed by our software [8, command `sm1.slope`, `slope`], which uses algorithms in [1], [2] and the set of the slopes is $\{-3/2\}$. Since $1/(1-s)$ agrees with $-3/2$, we have constructed a formal power series standing for the slope $-3/2$.

3 Gröbner Bases of Modified $\mathcal{A}$-Hypergeometric Systems

We will call $t = 0$ the *exceptional hypersurface* and we are interested in local analysis near $t = 0$. We denote by

$$\tau = (\mathbf{0}, -1; \mathbf{0}, 1)$$

the weight vector such that t has the weight -1 and ∂_t has the weight 1. θ_i is $x_i\partial_i$ and θ_t is $t\partial_t$. We denote by $\tilde{A}_{\theta,w,\beta}$ the set of the first $(d+1)$ Euler operators of the modified $\mathcal{A}$-hypergeometric system. For $\ell \in D$, $\mathrm{in}_\tau(\ell)$ is the sum of the maximal τ-order terms in ℓ. For a left ideal J in D, $\mathrm{in}_\tau(J)$ is the left ideal in D generated by $\mathrm{in}_\tau(\ell)$, $\ell \in J$ [11, p.4].

It is easy to see that, for generic w, $\mathrm{in}_\tau(D \cdot I_{\tilde{A}})$ can be generated by monomials in $\mathbf{C}[\partial_1, \ldots, \partial_n]$.

Theorem 2 *For generic β and w, we have*

$$\mathrm{in}_\tau(H_{A,w}(\beta)) = D \cdot \mathrm{in}_\tau(D \cdot I_{\tilde{A}}) + D \cdot \tilde{A}_{\theta,w,\beta} \tag{5}$$

Proof. Let $\mathcal{G}''$ be a Gröbner basis of the left ideal $H_{A,w}(\beta)$ for an order which refines the partial order determined by the weight vector τ. Then, the set $\{\mathrm{in}_\tau(g) \,|\, g \in \mathcal{G}''\}$ generates $\mathrm{in}_\tau(H_{A,w}(\beta))$ [11, Theorem 1.1.6]. The proof is performed by applying the Buchberger algorithm of computing the Gröbner basis $\mathcal{G}''$.

Let $s = (s_1, \ldots, s_d)$ be a vector of new indeterminates. Consider the algebra

$$D[s] = \mathbf{C}\langle x_1, \ldots, x_n, t, \partial_1, \ldots, \partial_n, \partial_t, s_1, \ldots, s_d\rangle$$

and its homogenized Weyl algebra by h

$$D[s]^h = \mathbf{C}\langle x_1, \ldots, x_n, t, \partial_1, \ldots, \partial_n, \partial_t, s_1, \ldots, s_d, h\rangle$$

where $\partial_i x_j = x_j \partial_i + h^2 \delta_{ij}$ and h commutes with other variables. Let H be the left ideal in $D[s]^h$ generated by $\tilde{A}_{\theta,w,s^2}$ and the homogenization of $I_{\tilde{A}}$. We define a partial order $>_\tau$ on monomials in $D[s]$ by

$$\begin{aligned} s^a x^b \partial^c t^d \partial_t^e >_\tau s^{a'} x^{b'} \partial^{c'} t^{d'} \partial_t^{e'} \quad \Leftrightarrow \quad & -d + e > -d' + e', \text{ or} \\ & -d + e = -d' + e' \text{ and } (a, e, d) >_{lex} (a', e', d') \end{aligned}$$

We refine this partial order by any monomial order and define orders $<$ in $D[s]$. (This order on $D[s]$ is extended to the order in the homogenized Weyl algebra and $D[s]^h$ as in [11, Section 1.2].)

Let $\mathcal{G}$ be the reduced Gröbner basis of the homogenized binomial ideal $I_{\tilde{A}}$ in $D[s]^h$ with respect to the order $<$. Note that the reduced Gröbner basis consists of elements of the form $\underline{\partial^u h^p} - \partial^v t^{v_{n+1}} h^{p'}$, $v_{n+1} > 0$ because w is generic and $I_{\tilde{A}}$ is saturated with respect to t. Note that either $p = 0$ or $p' = 0$ holds.

We will show that $\mathcal{G}$ and $\tilde{A}_{\theta,w,s^2}$ is a Gröbner basis $\mathcal{G}'$ with respect to $<$ in $D[s]^h$. This fact can be shown by checking the S-pair criterion in $D[s]^h$. It is easy to see that

$$sp(\underline{\theta_t} - \sum w_j \theta_j, \underline{s_i^2} - \sum a_{ij}\theta_j) \to_{\mathcal{G}'} 0$$

$$sp(\underline{s_k^2} - \sum a_{kj}\theta_j, \underline{s_i^2} - \sum a_{ij}\theta_j) \to_{\mathcal{G}'} 0$$

We assume $p > 0$ and $p' = 0$.

$$\begin{aligned} & sp(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}, \underline{s_i^2} - \sum a_{ij}\theta_j) \\ = \quad & s_i^2(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}) - \partial^u h^p(\underline{s_i^2} - \sum a_{ij}\theta_j)) \\ = \quad & -s_i^2 \partial^v t^{v_{n+1}} + \partial^u h^p \sum a_{ij}\theta_j \\ = \quad & -s_i^2 \partial^v t^{v_{n+1}} + \left(\sum a_{ij}\theta_j\right) \partial^u h^p + \left(\sum a_{ij} u_j\right) \partial^u h^p \end{aligned}$$

since $\partial^u h^p > \partial^v t^{v_{n+1}}$ we may rewrite it as

$$
\begin{aligned}
&= -s_i^2 \partial^v t^{v_{n+1}} + \left(\sum a_{ij}\theta_j\right)(\partial^u h^p - \partial^v t^{v_{n+1}}) + \left(\sum a_{ij}\theta_j\right)\partial^v t^{v_{n+1}} + \left(\sum a_{ij}u_j\right)\partial^u h^p \\
&= \left(\sum a_{ij}\theta_j\right)(\partial^u h^p - \partial^v t^{v_{n+1}}) + \partial^v t^{v_{n+1}}\left(\sum a_{ij}\theta_j - \sum a_{ij}v_j - s_i^2\right) + \left(\sum a_{ij}u_j\right)\partial^u h^p \\
&\quad \text{since } \sum a_{ij}u_j = \sum a_{ij}v_j \\
&= \left(\sum a_{ij}\theta_j\right)(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}) + \partial^v t^{v_{n+1}}\left(\sum a_{ij}\theta_j - s_i^2\right) + \left(\sum a_{ij}u_j\right)(\partial^u h^p - \partial^v t^{v_{n+1}}) \\
&\to_{\mathcal{G}'} 0
\end{aligned}
$$

The case $p = 0$, and $p' > 0$ can be shown analogously.

The final case we have to check is that

$$
\begin{aligned}
& sp(\underline{\partial^u h^p} - \partial^v t^{v_{n+1}}, \underline{\theta_t} - \sum w_j\theta_j) \\
&= -\theta_t \partial^v t^{v_{n+1}} + h^p \partial^u \sum w_j\theta_j \\
&= -\theta_t \partial^v t^{v_{n+1}} + \left(\sum w_j\theta_j + \sum w_j u_j\right) h^p \partial^u \\
&= -\theta_t \partial^v t^{v_{n+1}} + \left(\sum w_j\theta_j + \sum w_j u_j\right)(h^p\partial^u - \partial^v t^{v_{n+1}}) \\
&\qquad + \left(\sum w_j\theta_j + \sum w_j u_j\right)\partial^v t^{v_{n+1}} \\
&= -\theta_t \partial^v t^{v_{n+1}} + \left(\sum w_j\theta_j + \sum w_j u_j\right)(h^p\partial^u - \partial^v t^{v_{n+1}}) \\
&\qquad + \partial^v t^{v_{n+1}}\left(\sum w_j\theta_j + \sum w_j u_j - \sum w_j v_j\right) \\
&= \left(\sum w_j\theta_j + \sum w_j u_j\right)(h^p\partial^u - \partial^v t^{v_{n+1}}) \\
&\qquad + \partial^v t^{v_{n+1}}\left(\sum w_j\theta_j + \sum w_j u_j - \sum w_j v_j - \theta_t - v_{n+1}\right) \\
&= \left(\sum w_j\theta_j + \sum w_j u_j\right)\left(\underline{h^p\partial^u} - \partial^v t^{v_{n+1}}\right) + \partial^v t^{v_{n+1}}\left(\sum w_j\theta_j - \underline{\theta_t}\right) \\
&\to_{\mathcal{G}'} 0
\end{aligned}
$$

Specializing s to a vector of generic numbers β and h to 1 in $\mathcal{G}'$, we have a Gröbner basis of $H_{A,w}(\beta)$ with respect to τ. The correctness proof of this fact and the rest of the proof are analogous to [11, Theorem 3.1.3, p.106]. Q.E.D.

4 Indicial Polynomial along $t = 0$

The *indicial polynomial* of $H_{A,w}(\beta)$ along $t = 0$ is the monic generator $b(\theta_t)$ of the principal ideal $\mathrm{in}_\tau(H_{A,w}(\beta)) \cap \mathbf{C}[\theta_t]$. We fix a generic weight vector w. Since w is generic, $\mathrm{in}_\tau(I_{\tilde{A}})$ can be generated by monomials in $\mathbf{C}[\partial_1, \ldots, \partial_n]$. Let M be the monomial ideal generated by these monomials in $\mathbf{C}[\partial_1, \ldots, \partial_n]$. The top dimensional standard pairs are denoted by $\mathcal{T}(M)$ [11, p.112] and $\beta^{(\partial^\beta, \sigma)}$ is the zero point in $\mathbf{C}^n$ of the *distraction* $D \cdot M \cap \mathbf{C}[\theta_1, \ldots, \theta_n]$ of M and $A\theta - \beta$ associated to the standard pair (∂^β, σ) [11, p.68]. Here, the ring D in this paragraph is $\mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ and we regard the sub ring $\mathbf{C}[\theta_1, \ldots, \theta_n] \subset D$ as a commutative polynomial ring in variables θ_i.

Theorem 3 *Let both of β and w be generic. Then, the indicial polynomial $b(s)$ of $H_{A,w}(\beta)$ along $t = 0$ is*

$$\sum_{(\partial^\beta,\sigma)\in \mathcal{T}(M)} (s - w \cdot \beta^{(\partial^\beta,\sigma)}) \tag{6}$$

If $\mathcal{T}(M)$ is the empty set, the indicial polynomial is 0.

Proof. Under Theorem 2, the proof is analogous to [11, p.198, Proposition 5.1.9].

If the indicial polynomial is not constant and the difference of roots are not integral, we can construct formal series solution of the form

$$t^e \sum_{k=0}^{\infty} c_k(x) t^k, \quad c_k \in \mathbf{C}[1/x_1, \ldots, 1/x_n, x_1, \ldots, x_n] \tag{7}$$

where e is a root of the indicial polynomial and $t^e c_0(x)$ is a solution of the initial system $\mathrm{in}_{(\mathbf{0},-1;\mathbf{0},1)}(H_{A,w}(\beta))$. If the indicial polynomial is a constant, there is no formal series solution of the form above.

Example 2 (Continuation of Example 1). Note that $I_{\tilde{A}}$ is generated by $\underline{\partial_2} - t^3\partial_1^3$ and $\mathrm{in}_\tau(I_{\tilde{A}}) = \langle \partial_2 \rangle$. The distraction [11, p.68] of $M = \langle \partial_2 \rangle$ is θ_2. The zero set of the ideal generated by θ_2 and $A\theta - \beta = \{\theta_1 + 3\theta_2 + 1\}$ is $\{(-1, 0)\}$. Then, the indicial polynomial is $s - w \cdot (-1, 0) = s - 1$. The formal solution (4) stands for the root $s = 1$.

Example 3 Consider the modified hypergeometric system for $A = (-1, 1, 2)$, $\beta = (1/2)$, $w = (-2, -1, 0)$. This is the *Bessel function in two variables* called by Kimura and Okamoto (see, e.g., [6]).

The indicial polynomial is 1, because $I_{\tilde{A}} \ni \underline{1} - \partial_1^2\partial_3$. Then, there exists no series solution of the form (7). Incidentally, the set of the slopes along $t = 0$ at $x = (2, 2, 1)$ is equal to $\{-2, -3/2\}$. The values are obtained by our software [8].

Let us change w into $w = (3, 2, 1)$. The initial ideal $\mathrm{in}_\tau(I_{\tilde{A}})$ is generated by the set $\{\partial_1\partial_3, \partial_2^2, \partial_1\partial_2\}$ and the distraction of it is generated by $\theta_1\theta_3$, $\theta_2(\theta_2 - 1)$, $\theta_1\theta_2$. The set of zero points of the distraction and $A\theta - \beta$ is $\{(-1/2, 0, 0), (0, 0, 1/4), (0, 1, -1/4)\}$ which is obtained by computing the primary decomposition of the ideal generated by the distraction and $A\theta - \beta$ by the computer algebra system Risa/Asir [8] as

```
[0] load("gr"); load("primdec");
[107] [182] G=[x1*x3 , x2*(x2-1) ,  x1*x2 , -2*x1+2*x2+4*x3-1];
[x3*x1,x2^2-x2,x2*x1,-2*x1+4*x3+2*x2-1]
[183] primadec(G,[x1,x2,x3]);
[[[x3,x2,2*x1+1],[x3,x2,2*x1+1]],
 [[4*x3-1,x2,x1],[4*x3-1,x2,x1]],
 [[4*x3+1,x2-1,x1],[4*x3+1,x2-1,x1]]]
```

In this case, the generic condition for the Theorem 3 satisfied and the formula (6) gives the indicial polynomial $(s-3/2)(s+1/4)(s-7/4)$. We may use the `Risa/Asir` command `generic_bfct` to compute the indicial polynomial, but it uses a lot of memory and computation time when A is relatively large matrix. Our formula (6) can be applied to larger A. The rank of this system is 3. The set of the slopes along $t=0$ at $x=(2,2,1)$ is empty.

Although it is a side story in view of this paper, we want to note that a 3-dimensional graph of a solution of this system can be seen at `http://www.math.kobe-u.ac.jp/HOME/taka/test-bess2m.html`. We can see waves in two directions. This graph is drawn by series solutions and the Runge-Kutta method [6].

The number of solutions of the form (7) is less than the rank in general. It is an interesting open problem to construct rank many series solutions in terms of formal puiseux series and exponential functions along $t=0$.

Acknowledgments: This work was motivated by comments by Bernd Sturmfels in the joint study with Francisco Castro-Jimenez on slopes for $\mathcal{A}$-hypergeometric systems [2]. He asked "Is it possible to construct series solutions representing the slopes"? We realized that the original $\mathcal{A}$-hypergeometric system has few classical solutions representing slopes for some examples. The author introduced a modified system, which appears to be easier to analyze than the original system and may be the first step to studying the original $\mathcal{A}$-hypergeometric system. We did this work when Castro stayed in Japan in the spring of 2006. In fact, our Example 1 is an example to his question. However, we are still far from obtaining a complete answer. The author would like to thank both of them for all their comments and discussions.

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Finally, Go Okuyama posed a question on the lower bound of the rank of an $\mathcal{A}$-hypergeometric system. The Corollary 1 is the answer to his question.

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