Hierarchy of Bäcklund Transformation Groups of the Painlevé Systems *

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Abstract

For each Painlevé system P_J except the first one, we have a Bäcklund transformation group which is a lift of an affine Weyl group. In this paper, we show that the Bäcklund transformation groups for $J = V, IV, III, II$ are successively obtained from that for $J = VI$ by the well known degeneration or confluence processes.

1 Introduction

The J-th Painlevé system P_J $(J = VI, V, IV, III, II, I)$ which is equivalent to the J-th Painlevé equation is the following Hamiltonian system

$$P_J : \qquad \delta_J\, q = \{H_J(q,p,t,\alpha), q\}, \quad \delta_J\, p = \{H_J(q,p,t,\alpha), p\},$$

where $\delta_{VI} = t(t-1)d/dt$, $\delta_V = \delta_{III} = td/dt$, $\delta_{IV} = \delta_{II} = \delta_I = d/dt$, $\{\cdot,\cdot\}$ is a Poisson bracket defined by

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p},$$

and the Hamiltonian $H_J = H_J(q,p,t,\alpha)$ is of the form

$$\begin{aligned}
H_{VI}(q,p,t,\alpha) &= q(q-1)(q-t)p^2 - [(\alpha_0 - 1)q(q-1) + \alpha_4(q-1)(q-t) \\
&\qquad +\alpha_3 q(q-t)]p + \alpha_2(\alpha_1+\alpha_2)(q-t) \\
&\quad (\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4 = 1), \\
H_V(q,p,t,\alpha) &= q(q-1)p(p+t) - (\alpha_1+\alpha_3)qp + \alpha_1 p + \alpha_2 tq \\
&\quad (\alpha_0+\alpha_1+\alpha_2+\alpha_3 = 1), \\
H_{IV}(q,p,t,\alpha) &= qp(2p-q-2t) - 2\alpha_1 p - \alpha_2 q
\end{aligned}$$

*2000 Mathematics Subject Classification. Primary 34M55; Secondary 37K35. Key Words and Phrases. Painlevé systems, Hierarchy of Bäcklund transformation groups.

†The third author was partially supported by Grant-in-Aid for Scientific Research (No. 13440054 (b),(2)), Japan Society for the Promotion of Science.

$$
\begin{aligned}
& (\alpha_0+\alpha_1+\alpha_2=1), \\
H_{III}(q,p,t,\alpha) &= q^2p(p-1)+q[(\alpha_0+\alpha_2)p-\alpha_0]+tp \\
& (\alpha_0+2\alpha_1+\alpha_2=1), \\
H_{II}(q,p,t,\alpha) &= \frac{1}{2}p^2-(q^2+\frac{t}{2})p-\alpha_1 q \\
& (\alpha_0+\alpha_1=1), \\
H_I(q,p,t) &= \frac{1}{2}p^2-2q^3-tq.
\end{aligned}
$$

Notice that the Hamiltonian for $J = IV$ is slightly different from that in [4] but it is of the same form as in [1] and [9], becuase we use the well known degenerations in this paper.

The Bäcklund transformation group $W = W_J$ of Painlevé system P_J $(J \neq I)$ consists of birational symplectic transformations each of which preserves the form of the Hamiltonian H_J but changes the parameters $\alpha = (\alpha_0, ...)$ as an element of an affine Weyl group. In other words, the elements of W_J which is a lift of an affine Weyl group are Poisson bracket preserving differential isomorphisms of a differential field of functions of q, p, α equipped with a derivation defined by the system P_J and $\delta_J\,\alpha_i = 0$, $i = 0, 1, ...$ Here differential isomorphism means algebraic isomorphism commuting with the derivation. The group is generated by a finite set of generators $s_0, s_1, ...$ which correspond to the simple roots of the affine Lie algegra([5],[8]).

On the other hand, we know degenerations of Painlevé systems as the following diagram([1],[2],[8],[9]):

$$
\begin{array}{ccccccccc}
 & & & & P_{IV} & & & & \\
 & & & \nearrow & & \searrow & & & \\
P_{VI} & \longrightarrow & P_V & & & & P_{II} & \longrightarrow & P_I. \\
 & & & \searrow & & \nearrow & & & \\
 & & & & P_{III} & & & &
\end{array}
$$

For every $P_J \to P_K$ in the diagram, there is a change of parameters and variables

$$
\begin{aligned}
&\alpha_i = \alpha_i(A, \varepsilon) \quad (i = 0, 1, ...), \\
&t = t(\varepsilon, T), \quad q = q(A, \varepsilon, T, Q, P), \quad p = p(A, \varepsilon, T, Q, P),
\end{aligned}
$$

between $\alpha = (\alpha_0, \alpha_1, ...), t, q, p$ and $A = (A_0, A_1, ...), \varepsilon, T, Q, P$. For example, in the case of $P_{VI} \to P_V$,

$$
\begin{aligned}
&\alpha_0 = \varepsilon^{-1}, \quad \alpha_1 = A_3, \quad \alpha_2 = A_2, \quad \alpha_3 = A_0 - A_2 - \varepsilon^{-1}, \quad \alpha_4 = A_1, \\
&t = 1 + \varepsilon T, \quad (q-1)(Q-1) = 1, \quad (q-1)p + (Q-1)P = -A_2.
\end{aligned}
$$

Since the change of variables is symplectic, namely

$$\{P, Q\} = 1, \quad \{Q, Q\} = \{P, P\} = 0,$$

the system P_J is also written in the new variables T, Q, P and parameters A, ε as a Hamiltonian system denoted by $P_{J\to K}$. The system $P_{J\to K}$ tends to the system P_K as $\varepsilon \to 0$ and then the process $\varepsilon \to 0$ in the change of parameters and variables is called a degeneration or confluence process from P_J to P_K.

In this paper, we observe how the degeneration process from P_J to P_K works on the Bäcklund transformation group W_J. The change of parameters and variables lifts the group W_J to a group denoted again by W_J each element of which is a differential isomorphism of a differential field of functions of $A = (A_0, A_1, ...), \varepsilon, T, Q, P$. We see that an element of the new W_J does not converge as $\varepsilon \to 0$, in general. However we can verify the following theorem, which is the main assertion of this paper.

THEOREM. *For every degeneration process $P_J \to P_K$ except for $J = II, K = I$ in Painlevé systems, we can choose a subgroup $W_{J\to K}$ of the Bäcklund transformation group W_J so that $W_{J\to K}$ converges to W_K as $\varepsilon \to 0$.*

The subgroup $W_{J\to K}$ of W_J is taken as a group generated by reflections of $A_0, A_1, ...$, since the new parameters $A_0, A_1, ...$ should be the simple roots of an affine Weyl algebra for the system P_K.

Here we notice that the same process for $P_{II} \to P_I$ can be followed, however we see that each generator of $W_{II\to I}$ converges to the identity as $\varepsilon \to 0$. The fact seems to suggest that the first Painlevé system P_I has no nontrivial Bäcklund transformations.

Since each W_J is a lift of an affine Weyl group corresponding to an affine Lie algebra (see next section), it is convenient to express the above theorem by the following diagram:

$$\begin{array}{ccccccc} & & & & W(A_2^{(1)}) & & \\ & & & \nearrow & & \searrow & \\ W(D_4^{(1)}) & \longrightarrow & W(A_3^{(1)}) & & & & W(A_1^{(1)}). \\ & & & \searrow & & \nearrow & \\ & & & & W(C_2^{(1)}) & & \end{array}$$

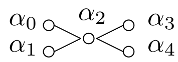
In Section 2, we review the Bäcklund transformation groups of the Painlevé systems P_J $(J \neq I)$. The following sections are devoted to the proof of the above theorem in all cases of degenerations. In these sections, we also see how $W_{J\to K}$ acts on the system $P_{J\to K}$.

2 Review of Bäcklund transformation groups

In this section, we give explicit forms of the generators s_i of the Bäcklund transformation group W of each Painlevé system. Each list consists of the type of affine Weyl group, Dynkin diagram, the fundamental relations of the generators of the group W, and the explicit forms of the generators, where the forms of $s_i(t)$ are omitted in the case of $s_i(t) = t$ for all i. The lists are the same as those in [4] except the case of $J = IV$.

2.1 The case of $J = VI$

$D_4^{(1)}$:



$(\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1)$

$$W(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4 \rangle : \quad s_i^2 = s_2^2 = 1, \quad (s_i s_j)^2 = 1, \quad (s_i s_2)^3 = 1.$$

	α_0	α_1	α_2	α_3	α_4	q	p
s_0	$-\alpha_0$	α_1	$\alpha_2 + \alpha_0$	α_3	α_4	q	$p - \frac{\alpha_0}{q-t}$
s_1	α_0	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	α_4	q	p
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$\alpha_4 + \alpha_2$	$q + \frac{\alpha_2}{p}$	p
s_3	α_0	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	α_4	q	$p - \frac{\alpha_3}{q-1}$
s_4	α_0	α_1	$\alpha_2 + \alpha_4$	α_3	$-\alpha_4$	q	$p - \frac{\alpha_4}{q}$

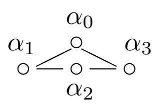
The last list should be read as

$$s_0(\alpha_0) = -\alpha_0, \quad s_0(\alpha_1) = \alpha_1, \quad s_0(\alpha_2) = \alpha_2 + \alpha_0, \quad s_0(\alpha_3) = \alpha_3, \quad s_0(\alpha_4) = \alpha_4,$$
$$s_0(q) = q, \quad s_0(p) = p - \frac{\alpha_0}{q - t}$$

and so on.

2.2 The case of $J = V$

$A_3^{(1)}$:



$(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1)$

$$W(A_3^{(1)}) = \langle s_0, s_1, s_2, s_3 \rangle : \quad s_i^2 = 1, \quad (s_i s_{i+2})^2 = 1, \quad (s_i s_{i+1})^3 = 1.$$

	α_0	α_1	α_2	α_3	q	p
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	α_2	$\alpha_3 + \alpha_0$	$q + \frac{\alpha_0}{p+t}$	p
s_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	q	$p - \frac{\alpha_1}{q}$
s_2	α_0	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$q + \frac{\alpha_2}{p}$	p
s_3	$\alpha_0 + \alpha_3$	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	q	$p - \frac{\alpha_3}{q-1}$

2.3 The case of $J = IV$

$$A_2^{(1)}: \quad \begin{array}{ccc} & \overset{\alpha_0}{\circ} & \\ \overset{\alpha_1}{\circ} & & \overset{\alpha_2}{\circ} \end{array} \qquad (\alpha_0 + \alpha_1 + \alpha_2 = 1)$$

$$W(A_2^{(1)}) = \langle s_0, s_1, s_2 \rangle : \quad s_0^2 = s_1^2 = s_2^2 = 1, \quad (s_0 s_1)^3 = (s_1 s_2)^3 = (s_2 s_0)^3 = 1.$$

	α_0	α_1	α_2	q	p
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$	$q + \frac{2\alpha_0}{2p-q-2t}$	$p + \frac{\alpha_0}{2p-q-2t}$
s_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	q	$p - \frac{\alpha_1}{q}$
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$q + \frac{\alpha_2}{p}$	p

2.4 The case of $J = III$

$$C_2^{(1)}: \quad \overset{\alpha_0}{\circ} \Rightarrow \overset{\alpha_1}{\circ} \Leftarrow \overset{\alpha_2}{\circ} \qquad (\alpha_0 + 2\alpha_1 + \alpha_2 = 1)$$

$$W(C_2^{(1)}) = \langle s_0, s_1, s_2 \rangle : \quad s_0^2 = s_1^2 = s_2^2 = 1, \quad (s_0 s_1)^4 = (s_1 s_2)^4 = 1.$$

	α_0	α_1	α_2	t	q	p
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	α_2	t	$q + \frac{\alpha_0}{p}$	p
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$\alpha_2 + 2\alpha_1$	$-t$	q	$p - \frac{2\alpha_1}{q} + \frac{t}{q^2}$
s_2	α_0	$\alpha_1 + \alpha_2$	$-\alpha_2$	t	$q + \frac{\alpha_2}{p-1}$	p

2.5 The case of $J = II$

$$A_1^{(1)}: \quad \overset{\alpha_0}{\circ} \Leftrightarrow \overset{\alpha_1}{\circ} \qquad (\alpha_0 + \alpha_1 = 1)$$

$$W(A_1^{(1)}) = \langle s_0, s_1 \rangle : \quad s_0^2 = s_1^2 = 1.$$

	α_0	α_1	q	p
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$q + \frac{\alpha_0}{p-2q^2-t}$	$p + \frac{4\alpha_0 q}{p-2q^2-t} + \frac{2\alpha_0^2}{(p-2q^2-t)^2}$
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$q + \frac{\alpha_1}{p}$	p

3 Degeneration from W_{VI} to W_V

In this case, the degeneration process is given by

$$\alpha_0 = \varepsilon^{-1}, \quad \alpha_1 = A_3, \quad \alpha_2 = A_2, \quad \alpha_3 = A_0 - A_2 - \varepsilon^{-1}, \quad \alpha_4 = A_1, \tag{3.1}$$

$$t = 1 + \varepsilon T, \quad (q-1)(Q-1) = 1, \quad (q-1)p + (Q-1)P = -A_2. \tag{3.2}$$

Notice that $A_0 + A_1 + A_2 + A_3 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$ and the change of variables from (q,p) to (Q,P) is symplectic.

Each Bäcklund transformation in W_{VI} given in **2.1** is an differential isomorphism of the differential field $K = \mathbf{C}(\alpha, t, q, p)$ of rational functions of $\alpha = (\alpha_0, \alpha_1, ..., \alpha_4), t, q, p$ equipped with a derivation δ_{VI} defined by

$$\delta_{VI}\, q = \{H_{VI}, q\}, \quad \delta_{VI}\, p = \{H_{VI}, p\},$$
$$\delta_{VI}\, t = t(t-1), \quad \delta_{VI}\, \alpha_i = 0, \ i = 0, 1, ..., 4.$$

Since the change of parameters and variables (3.1),(3.2) is birational, we can obtain the action of W_{VI} on the differential field $K' := \mathbf{C}(A, \varepsilon, T, Q, P)$ of rational functions of $A = (A_0, A_1, A_2, A_3), \varepsilon, T, Q, P$.

Let us see the actions of the generators $s_i,\ i = 0, 1, 2, 3, 4$ on the parameters $A_i,\ i = 0, 1, 2, 3$ and ε where

$$A_0 = \alpha_0 + \alpha_2 + \alpha_3, \quad A_1 = \alpha_4, \quad A_2 = \alpha_2, \quad A_3 = \alpha_1 \quad \varepsilon = \frac{1}{\alpha_0}.$$

For example, the action of s_0 is obtained as

$$\begin{aligned}
&s_0(A_0) = s_0(\alpha_0 + \alpha_2 + \alpha_3) = -\alpha_0 + (\alpha_2 + \alpha_0) + \alpha_3 = \alpha_2 + \alpha_3 \\
&\quad = A_0 - \varepsilon^{-1}, \quad s_0(A_1) = s_0(\alpha_4) = \alpha_4 = A_1, \\
&s_0(A_2) = s_0(\alpha_2) = \alpha_2 + \alpha_0 = A_2 + \varepsilon^{-1}, \quad s_0(A_3) = s_0(\alpha_1) = \alpha_1 = A_3, \\
&s_0(\varepsilon) = s_0(1/\alpha_0) = -1/\alpha_0 = -\varepsilon.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&s_1(A_0) = A_0 + A_3,\ s_1(A_1) = A_1,\ s_1(A_2) = A_2 + A_3,\ s_1(A_3) = -A_3,\ s_1(\varepsilon) = \varepsilon \\
&s_2(A_0) = A_0,\ s_2(A_1) = A_1 + A_2,\ s_2(A_2) = -A_2,\ s_2(A_3) = A_3 + A_2, \\
&s_2(\varepsilon) = \frac{\varepsilon}{1 + A_2\varepsilon}, \\
&s_3(A_0) = A_2 + \varepsilon^{-1},\ s_3(A_1) = A_1,\ s_3(A_2) = A_0 - \varepsilon^{-1},\ s_3(A_3) = A_3,\ s_3(\varepsilon) = \varepsilon, \\
&s_4(A_0) = A_0 + A_1,\ s_4(A_1) = -A_1,\ s_4(A_2) = A_2 + A_1,\ s_4(A_3) = A_3,\ s_3(\varepsilon) = \varepsilon.
\end{aligned}$$

We remark that $s_3(A_0)$ and $s_3(A_2)$ diverge as $\varepsilon \to 0$.

Obersving these relations, we take a subgroup $W_{VI\to V}$ of W_{VI} genetated by S_0, S_1, S_2, S_3 defined by

$$S_0 := s_0 s_2 s_3 s_2 s_0 = s_3 s_2 s_0 s_2 s_3, \quad S_1 := s_4, \quad S_2 := s_2, \quad S_3 := s_1. \tag{3.3}$$

We can easily check

$$
\begin{aligned}
&(3.4) && S_0(A_0) = -A_0, \quad S_0(A_1) = A_1 + A_0, \quad S_0(A_2) = A_2, \\
& && S_0(A_3) = A_3 + A_0, \quad S_0(\varepsilon) = \frac{\varepsilon}{1 - A_2\varepsilon}, \\
&(3.5) && S_1(A_0) = A_0 + A_1, \quad S_1(A_1) = -A_1, \quad S_1(A_2) = A_2 + A_1, \\
& && S_1(A_3) = A_3, \quad S_1(\varepsilon) = \varepsilon, \\
&(3.6) && S_2(A_0) = A_0, \quad S_2(A_1) = A_1 + A_2, \quad S_2(A_2) = -A_2, \\
& && S_2(A_3) = A_3 + A_2, \quad S_2(\varepsilon) = \frac{\varepsilon}{1 + A_2\varepsilon}, \\
&(3.7) && S_3(A_0) = A_0 + A_3, \quad S_3(A_1) = A_1, \quad S_3(A_2) = A_2 + A_3, \\
& && S_3(A_3) = -A_3, \quad S_3(\varepsilon) = \varepsilon,
\end{aligned}
$$

and the generators satisfy the fundamental relations given in **2.2**. In short, the group $W_{VI \to V} =< S_0, S_1, S_2, S_3 >$ can be considered to be an affine Weyl group of the affine Lie algebra of type $A_3^{(1)}$ with simple roots A_0, A_1, A_2, A_3.

Now we investigate how the generators of $W_{VI \to V}$ act on T, Q and P. We can verify

$$
\begin{aligned}
&(3.8) && S_0(T) = T(1 - A_0\varepsilon), \quad S_0(Q) = Q + \frac{A_0(1 - Q(Q-1)P\varepsilon)}{P + T - T(Q-1)P\varepsilon}, \\
& && S_0(P) = P\Big(1 + \frac{A_0 T\varepsilon}{P + T - T(A_0 + QP)\varepsilon}\Big), \\
&(3.9) && S_1(T) = T, \quad S_1(Q) = Q, \quad S_1(P) = P - \frac{A_1}{Q}, \\
&(3.10) && S_2(T) = T(1 + A_2\varepsilon), \quad S_2(Q) = Q + \frac{A_2}{P}, \quad S_2(P) = P, \\
&(3.11) && S_3(T) = T, \quad S_3(Q) = Q, \quad S_3(P) = P - \frac{A_3}{Q-1}.
\end{aligned}
$$

By comparing (3.4) – (3.11) with the last list in **2.2**, we see that our theorem holds for $W_{VI} \to W_V$.

We notice that the system P_{VI} is written in the new variables as

$$
P_{VI \to V}: \qquad \delta_V\, Q = \{H_{VI \to V}, Q\}, \quad \delta_V\, P = \{H_{VI \to V}, P\}
$$

where $\delta_V = T\partial/\partial T$, $H_{VI \to V} := H_{VI}/(1 + \epsilon T)$, $H_{VI \to V} \to H_V$ as $\varepsilon \to 0$. We can verify that δ_V commutes with any element $W_{VI \to V}$, and then for any $w \in W_{VI \to V}$

$$
\delta_V\, w(Q) = \{w(H_{VI \to V}), w(Q)\}, \quad \delta_V\, w(P) = \{w(H_{VI \to V}), w(P)\}.
$$

4 Degeneration from W_V to W_{IV}

The degeneration in the case is given by

$$\alpha_0 = A_0 + \frac{1}{2}\varepsilon^{-2}, \quad \alpha_1 = A_1, \quad \alpha_2 = A_2, \quad \alpha_3 = -\frac{1}{2}\varepsilon^{-2}, \tag{4.1}$$

$$t = \frac{1}{2}\varepsilon^{-2}(1+2\varepsilon T), \quad q = -\frac{\varepsilon Q}{1-\varepsilon Q}, \tag{4.2}$$
$$p = -\varepsilon^{-1}(1-\varepsilon Q)[P - \varepsilon(A_2 + QP)].$$

Notice that $A_0 + A_1 + A_2 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$ and the transformation from (q, p) to (Q, P) is symplectic, however the change of parameters (4.1) is not one to one differently from the case of $P_{VI} \to P_V$.

Since the generators of $W_{V\to IV}$ should be reflections of $A_0 = \alpha_0 + \alpha_3, A_1 = \alpha_1, A_2 = \alpha_2$, we choose them as

$$S_0 := s_3 s_0 s_3 = s_0 s_3 s_0, \quad S_1 := s_1, \quad S_2 := s_2 \tag{4.3}$$

and set $W_{V\to VI} =< S_0, S_1, S_2 >$. Then we immediately have

$$S_0(A_0) = -A_0, \quad S_0(A_1) = A_1 + A_0, \quad S_0(A_2) = A_2 + A_0, \tag{4.4}$$

$$S_1(A_0) = A_0 + A_1, \quad S_1(A_1) = -A_1, \quad S_1(A_2) = A_2 + A_1, \tag{4.5}$$

$$S_2(A_0) = A_0 + A_2, \quad S_2(A_1) = A_1 + A_2, \quad S_2(A_2) = -A_2. \tag{4.6}$$

However, we see that $S_i(\varepsilon)$ have ambiguities of signature. For example, since

$$S_2(\varepsilon)^2 = s_2(\varepsilon^2) = s_2((-1/2)/\alpha_3) = -\frac{1}{2}\frac{1}{\alpha_3 + \alpha_2} = \frac{\varepsilon^2}{1 - 2A_2\varepsilon^2},$$

we can choose any one of the two branches as $S_2(\varepsilon)$. Among such possibilities, we take a choice as

$$S_0(\varepsilon) = \varepsilon(1 + 2A_0\varepsilon^2)^{-1/2}, \quad S_1(\varepsilon) = \varepsilon, \quad S_2(\varepsilon) = \varepsilon(1 - 2A_2\varepsilon^2)^{-1/2} \tag{4.7}$$

where $(1 + 2A_0\varepsilon^2)^{1/2} = 1$ and $(1 - 2A_2\varepsilon^2)^{1/2} = 1$ at $A_0\varepsilon^2 = 0$ and $A_2\varepsilon^2 = 0$ respectively, or considering in the category of formal power series, we make a convention that $(1 + 2A_0\varepsilon^2)^{1/2}$ and $(1 - 2A_2\varepsilon^2)^{1/2}$ are formal power series of $A_0\varepsilon^2$ and $A_2\varepsilon^2$ with constant terms 1 according to

$$(1 + x)^c \sim 1 + \sum_{n\geq 1} \binom{c}{n} x^n.$$

We notice that the generators acting on parameters $A_0, A_1, A_2, \varepsilon$ satisfy the fundamental relations in **2.3**.

Now we observe the actions of $S_i, i = 0, 1, 2$ on the variables T, Q, P. By means of (4.2),(4.7) and

$$S_0(t) = s_3 s_0 s_3(t) = t, \quad S_1(t) = s_1(t) = t, \quad S_2(t) = s_2(t) = t,$$

we can easily check

$$S_0(T) = (T - A_0\varepsilon)(1 + 2A_0\varepsilon^2)^{-1/2}, \quad S_1(T) = T, \tag{4.8}$$

$$S_2(T) = (T + A_2\varepsilon)(1 - 2A_2\varepsilon^2)^{-1/2}. \tag{4.9}$$

By (4.1),(4.2),(4.7) and the actions of s_1, s_2 on q, p, we can easily verify

$$S_1(Q) = Q, \quad S_1(P) = P - \frac{A_1}{Q} \tag{4.10}$$

$$S_2(Q) = Q + \frac{A_2}{P}, \quad S_2(P) = P. \tag{4.11}$$

The forms of the actions $S_0 = s_3 s_0 s_3$ on Q and P are complicated, but we can see that

$$S_0(Q) \to Q + \frac{2A_0}{2P - Q - 2T}, \quad S_0(P) \to P + \frac{A_0}{2P - Q - 2T} \tag{4.12}$$

as $\varepsilon \to 0$ for arbitrarily fixed $A = (A_0, A_1, A_2), T, Q$ and P with some generic conditions such as $2P - Q - 2T \neq 0$. Here we have to note that, although $S_0(Q), S_0(P)$ contain formal power series of A, ε, they are analytic if ε is sufficiently small for any fixed A, T, Q, P.

By means of the above study, we define a differential field K' on which $W_{V\to IV} =< S_0, S_1, S_2 >$ acts as the field of rational functions of T, Q, P whose coefficients are formal power series of $A_0, A_1, A_2, \varepsilon$. Then the action of any $w \in W_{V\to IV}$ is defined as an isomorphism from K' to itself.

The equations or property from (4.4) to (4.12) and the list in **2.3** show the theorem for $W_V \to W_{IV}$.

Since $\delta_V = td/dt = (1 + 2\varepsilon T)(2\varepsilon)^{-1} d/dT = (1 + 2\varepsilon T)(2\varepsilon)^{-1}\delta_{IV}$ and the transformation from (q, p) to (Q, P) is symplectic, the system P_V is expressed as

$$P_{V\to IV}: \qquad \delta_{IV} Q = \{H_{V\to IV}, Q\}, \quad \delta_{IV} P = \{H_{V\to IV}, P\}$$

in the new variables, where $H_{V\to IV} = 2\varepsilon(1 + 2\varepsilon T)^{-1} H_V$, and $H_{V\to IV} \to H_{IV}$ as $\varepsilon \to 0$. However δ_{IV} does not commutes with the elements of $W_{V\to IV}$ and then we have to notice that the transform of $P_{V\to IV}$ by $w \in W_{V\to IV}$ is

$$\delta_{IV} w(Q) = \Big\{\frac{2\varepsilon}{1 + 2\varepsilon T} w\Big(\frac{1 + 2\varepsilon T}{2\varepsilon}\Big) w(H_{V\to IV}), w(Q)\Big\},$$
$$\delta_{IV} w(P) = \Big\{\frac{2\varepsilon}{1 + 2\varepsilon T} w\Big(\frac{1 + 2\varepsilon T}{2\varepsilon}\Big) w(H_{V\to IV}), w(P)\Big\},$$

which is verified by the fact that δ_V commutes with every $w \in W_{V\to IV}$.

5 Degeneration from W_V to W_{III}

The degeneration in this case is

$$\alpha_0 = A_2, \quad \alpha_1 = \varepsilon^{-1}, \quad \alpha_2 = A_0, \quad \alpha_3 = 2A_1 - \varepsilon^{-1}, \tag{5.1}$$

$$t = -\varepsilon T, \quad q = 1 + \frac{Q}{\varepsilon T}, \quad p = \varepsilon T P. \tag{5.2}$$

We see that $A_0 + 2A_1 + A_2 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$ and the change of variables from (q, p) to (Q, P) is symplectic. As the case of $P_{VI} \to P_V$, the transformation given by (5.1) and (5.2) is birational, and we can easily obtain the actions of s_i, $i = 0, 1, 2, 3$ on the differential field $K' = \mathbf{C}(A_0, A_1, A_2, \varepsilon, T, Q, P)$.

Choose S_i, $i = 0, 1, 2$ as

$$S_0 := s_2, \quad S_1 := s_3 s_1 = s_1 s_3, \quad S_2 := s_0 \tag{5.3}$$

which are reflections of $A_0 = \alpha_2$, $A_1 = (\alpha_1 + \alpha_3)/2$, $A_2 = \alpha_0$ respectively.

It is easy to see that

$$S_0(A_0) = -A_0,\ S_0(A_1) = A_1 + A_0,\ S_0(A_2) = A_2,\ S_0(\varepsilon) = \frac{\varepsilon}{1 + A_0\varepsilon} \tag{5.4}$$

$$S_1(A_0) = A_0 + 2A_1,\ S_1(A_1) = -A_1\ S_1(A_2) = A_2 + 2A_1,\ S_1(\varepsilon) = -\varepsilon, \tag{5.5}$$

$$S_2(A_0) = A_0,\ S_2(A_1) = A_1 + A_2,\ S_2(A_2) = -A_2,\ S_2(\varepsilon) = \frac{\varepsilon}{1 + A_2\varepsilon} \tag{5.6}$$

and

$$S_0(T) = T(1 + A_0\varepsilon), \quad S_0(Q) = Q + \frac{A_0}{P}, \quad S_0(P) = P, \tag{5.7}$$

$$S_1(T) = -T, \quad S_1(Q) = Q, \quad S_1(P) = P - \frac{2A_1}{Q} + \frac{T}{Q^2} + O(\varepsilon), \tag{5.8}$$

$$S_2(T) = T(1 + A_2\varepsilon), \quad S_2(Q) = Q + \frac{A_2}{P - 1}, \quad S_2(P) = P \tag{5.9}$$

where $O(\varepsilon)$ is a rational function of $A_i, i = 0, 1, 2,\ \varepsilon, T, Q, P$ with a factor ε. The proof of the theorem for $W_V \to W_{III}$ has thus been completed.

We see that $\delta_V = td/dt = Td/dT = \delta_{III}$ and the system P_V is written in the new variables by

$$P_{V\to III}: \qquad \delta_{III} Q = \{H_{V\to III}, Q\}, \quad \delta_{III} P = \{H_{V\to III}, P\}$$

where $H_{V\to III} = H_V + QP$, which converges to H_{III} as $\varepsilon \to 0$. Since δ_{III} commutes with any element of $W_{V\to III}$, the transform of $P_{V\to III}$ by $w \in W_{V\to III}$ is

$$\delta_{III} w(Q) = \{w(H_{V\to III}), w(Q)\}, \quad \delta_{III} w(P) = \{w(H_{V\to III}), w(P)\}.$$

6 Degeneration from W_{IV} to W_{II}

The degeneration is

$$\alpha_0 = A_0 - \frac{1}{4}\varepsilon^{-6}, \quad \alpha_1 = \frac{1}{4}\varepsilon^{-6}, \quad \alpha_2 = A_1, \tag{6.1}$$

$$t = -\frac{1}{\sqrt{2}}\varepsilon^{-3}(1-\varepsilon^4 T), \quad q = \frac{1}{\sqrt{2}}\varepsilon^{-3}(1+2\varepsilon^2 Q), \quad p = \frac{1}{\sqrt{2}}\varepsilon P. \tag{6.2}$$

Then $A_0 + A_1 = \alpha_0 + \alpha_1 + \alpha_2 = 1$ and the change of variables from (q, p) to (Q, P) is symplectic. Since the change of parameters (6.1) is not one to one, we consider the degeneration process by introducing formal power series of the new parameters $A = (A_0, A_1), \varepsilon$.

We choose S_0 and S_1 as

$$S_0 := s_0 s_1 s_0 = s_1 s_0 s_1, \quad S_1 := s_2 \tag{6.3}$$

and put $W_{IV\to II} =< S_0, S_1 >$. Note that S_0, S_1 are reflections of $A_0 = \alpha_0 + \alpha_1$, $A_1 = \alpha_2$ respectively.

Then we can obtain

$$S_0(A_0) = -A_0, \quad S_0(A_1) = A_1 + 2A_0, \quad S_0(\varepsilon) = \varepsilon(1 - 4A_0\varepsilon^6)^{-1/6}, \tag{6.4}$$

$$S_1(A_0) = A_0 + 2A_1, \quad S_1(A_1) = -A_1, \quad S_1(\varepsilon) = \varepsilon(1 + 4A_1\varepsilon^6)^{-1/6}. \tag{6.5}$$

Here, we make the same convention as in Section 4 that $(1 - 4A_0\varepsilon^6)^{-1/6}$ and $(1 + 4A_1\varepsilon^6)^{-1/6}$ respectively mean formal power series of $A_0\varepsilon^6$ and $A_1\varepsilon^6$ with 1 as constant terms.

Let K' be a field of rational functions of T, Q, P whose coefficients are formal power series of $A = (A_0, A_1), \varepsilon$. Then we can verify

$$\begin{aligned} &S_0(T) \to T, \quad S_0(Q) \to Q + \frac{A_0}{P - 2Q^2 - T}, \\ &S_0(P) \to P + \frac{4A_0 Q}{P - 2Q^2 - T} + \frac{2A_0^2}{(P - 2Q^2 - T)^2}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} &S_1(T) \to T, \quad S_1(Q) \to Q + \frac{A_1}{P}, \\ &S_1(P) \to P \end{aligned} \tag{6.7}$$

as $\varepsilon \to 0$. Concerning the convergence, remind the note in Section 4. Thus we have proved the theorem for $W_{IV} \to W_{II}$.

Since $\delta_{IV} = (\sqrt{2}/\varepsilon)\delta_{II}$, the system P_{IV} is written in the new variables as

$$P_{IV\to II}: \qquad \delta_{II} Q = \{H_{IV\to II}, Q\}, \quad \delta_{II} P = \{H_{IV\to II}, P\}$$

where $H_{IV\to II} = (\varepsilon/\sqrt{2})H_{IV}$ and $H_{IV\to II} \to H_{II}$ as $\varepsilon \to 0$. Notice that δ_{II} does not commute with elements of $W_{IV\to II}$, and the transform of $P_{IV\to II}$ by

$w \in W_{IV\to II}$ is

$$\delta_{II} w(Q) = \{\varepsilon w(1/\varepsilon) w(H_{IV\to II}), w(Q)\},$$
$$\delta_{II} w(P) = \{\varepsilon w(1/\varepsilon) w(H_{IV\to II}), w(P)\}.$$

7 Degeneration from W_{III} to W_{II}

In this case, the degeneration of parameters is given by

$$\alpha_0 = A_1, \quad \alpha_1 = \frac{1}{4}\varepsilon^{-3}, \quad \alpha_2 = A_0 - \frac{1}{2}\varepsilon^{-3} \tag{7.1}$$

and that of variables is given by the composition of the following two transformations:

$$t = -\tau^2, \quad q = -\frac{\tau}{x}, \quad p = \frac{x}{\tau}(A_1 + xy), \tag{7.2}$$

$$\tau = \frac{1+\varepsilon^2 T}{4\varepsilon^3}, \quad x = 1 + 2\varepsilon Q, \quad y = \frac{P}{2\varepsilon}. \tag{7.3}$$

Note that $A_0 + A_1 = \alpha_0 + 2\alpha_1 + \alpha_2 = 1$ and the transformations from (q, p) to (x, y) and from (x, y) to (Q, P) are symplectic.

Let us choose

$$S_0 := (s_2 s_1)^2 = (s_1 s_2)^2, \quad S_1 := s_0 \tag{7.4}$$

as generators of $W_{III\to II}$. Then we see that

$$S_0(A_0) = -A_0, \quad S_0(A_1) = A_1 + 2A_0, \quad S_0(\varepsilon) = -\varepsilon, \tag{7.5}$$

$$S_1(A_0) = A_0 + 2A_1, \quad S_1(A_1) = -A_1, \quad S_1(\varepsilon) = \varepsilon(1 + 4A_1\varepsilon^3)^{-1/3}. \tag{7.6}$$

In the last equation of (7.5), we have chosen -1 as a branch of $(-1)^{1/3}$ in order that $S_0^2(\varepsilon) = \varepsilon$. As in Sections 4,6, we make a convention that $(1 + 4A_1\varepsilon^3)^{-1/3}$ is a formal power series of $A_1\varepsilon^3$ with 1 as a constant term.

By careful calculation, we can verify

$$S_0(T) = T, \quad S_0(Q) \to Q + \frac{A_0}{P - 2Q^2 - T} \tag{7.7}$$

$$S_0(P) \to P + \frac{4A_0 Q}{P - 2Q^2 - T} + \frac{2A_0^2}{(P - 2Q^2 - T)^2},$$

$$S_1(T) \to T, \quad S_1(Q) \to Q + \frac{A_1}{P}, \quad S_1(P) \to P \tag{7.8}$$

as $\varepsilon \to 0$ for arbitrarily fixed A_0, A_1, T, Q, T. Thus we have proved the theorem for $W_{III} \to W_{II}$.

We see that $\delta_{III} = (1+\varepsilon^2 T)(2\varepsilon^2)^{-1}\delta_{II}$ and the system P_{III} is written in the new variables as

$$P_{III\to II}: \qquad \delta_{II}Q = \{H_{III\to II}, Q\}, \quad \delta_{II}P = \{H_{III\to II}, P\}$$

where $H_{III\to II} = (2\varepsilon^2)(1+\varepsilon^2 T)^{-1}H_{III}$ and $H_{III\to II} \to H_{II}$ as $\varepsilon \to 0$. We notice that δ_{II} does not commute with elements of $W_{III\to II}$, and the transform of $P_{IV\to II}$ by $w \in W_{III\to II}$ is

$$\delta_{II}w(Q) = \Big\{\frac{2\varepsilon^2}{1+\varepsilon^2 T}w\Big(\frac{1+\varepsilon^2 T}{2\varepsilon^2}\Big)w(H_{III\to II}), w(Q)\Big\},$$
$$\delta_{IV}w(P) = \Big\{\frac{2\varepsilon^2}{1+\varepsilon^2 T}w\Big(\frac{1+\varepsilon^2 T}{2\varepsilon^2}\Big)w(H_{III\to II}), w(P)\Big\}.$$

Acknowledgement. The problem studied in this paper was proposed by Prof. Kenji Iohara in Kobe University. The authors thank him not only for the proposal but also for helpful discussions.

References

1. K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé, Vieweg, 1991.
2. T. Masuda, On a class of algebraic solutions to the Painlevé VI equation, its determinant formula and coalescence cascade, Funkcial. Ekvac., **46**(2003), 121–171.
3. M. Noumi, Painlevé equations, Asakura Shoten, Tokyo, 2000 (in Japanese).
4. M. Noumi, K. Takano and Y. Yamada, Bäcklund transformations and the manifolds of Painlevé systems, Funkcial. Ekvac., **45** (2002), 237–258.
5. M. Noumi and Y. Yamada, Affine Weyl groups, discrete dynamical systems and Painlevé equations, Commun. Math. Phys., **199**(1998), 281–295.
6. M. Noumi and Y. Yamada, Affine Weyl group symmetries in Painlevé type equations, in Toward the exact WKB analysis of differential equations, linear or non-linear (Eds. C.J. Howl, T. Kawai, Y. Takei), 245–259, Kyoto Univ. Press, Kyoto, 2000.
7. K. Okamoto, Studies on the Painlevé equations, I, Ann. Mat. Pura Appl., **146**(1987), 337-381; II, Jap. J. Math., **13**(1987), 47-76; III, Math. Ann., **275**(1986), 221-256; IV, Funkcial. Ekvac., **30** (1987), 305-332.
8. P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixes, C. R. Acad. Sci. Paris, **143**(1906), 1111–1117.

9. K. Takano, Confluence processes in defining manifolds for Painlevé systems, Tohoku Math. J., **53**(2001), 319–335.

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