Defining Manifolds for Painlvé Equations

Kyoichi TAKANO (Kobe Univ.)

1 Manifolds constructed by Okamoto

Each Painlevé equation is equivalent to a polynomial Hamiltonian system, namely, the J-th Painlevé equation is equivalent to a Hamiltonian system (which we call the J-th Painlevé system)

$$dx/dt = \partial H_J/\partial y, \quad dy/dt = -\,\partial H_J/\partial x,$$

where $H_J = H_J(x, y, t)$ is a polynomial of x and y of which the coefficients are rational functions of t holomorphic in $B_J := \mathbf{P} - \Xi_J$, $\Xi_J \ni \infty$ being the fixed singular points of the J-th Painlevé equation (Ref. 2). Here the equivalece means that the second order nonlinear differential equation in x obtained from the J-th Painlevé system by elimination of y is just the J-th Painlevé equation.

The J-th Painlevé system defines a nonsingular foliation of the trivial fiber space $(\mathbf{C}^2 \times B_J, \pi_J, B_J)$ every leaf of which is transversal to fibers. However, the foliation is not *uniform*. (Uniformity means that *for any point* $P_0 \in \mathbf{C}^2 \times B_J$*, every curve in* B_J *with starting point* $\pi_J(P_0)$ *can be lifted on the leaf passing through the point* P_0.)

In the paper Ref. 5, K. Okamoto constructed, for each J, a fiber space (E_J, π_J, B_J) having the $(\mathbf{C}^2 \times B_J, \pi_J, B_J)$ as a fiber subspace so that the extension of the J-th Painlevé system defines a *uniform* foliation of E_J. The uniformity of the foliation is equivalent to the so-called *Painlevé property* which is stated as: *for any* $(x_0, y_0, t_0) \in \mathbf{C}^2 \times B_J$*,* $x(t)$ *and* $y(t)$ *can be meromorphically continued along any curve in* B_J *with starting point* t_0*, where* $(x(t), y(t))$ *is the local solution of the* J*-th Painlevé system satisfying the initial condition* $(x(t_0), y(t_0)) = (x_0, y_0)$. Okamoto called each fiber $E_J(t) := \pi_J^{-1}(t)$ the *space of initial conditions*, because there is a bijection from $E_J(t)$ to the set of all solutions of the J-th Painlevé system, however he did not name the total space E_J itself. So I call it *defining manifold* in this note.

Okamoto constructed each fiber $E_J(t)$ $(t \in B_J)$ by compactification of $\mathbf{C}^2 \times t$, 8 times quadratic transformations (the result is denoted by

$\overline{E}(t)$), and removing of divisors of self-intersection number -2. Then he obtained the defining manifold $E_J = \bigsqcup_{t \in B_J} E_J(t)$.

The manifold E_J ($J = VI, ..., II$) is described as a patching of several charts $\{V_J(*) = \mathbf{C}^2 \times B_J\}_*$ by *rational* and *symplectic* coordinate transformations (Ref. 9, 10), and then the J-th Painlevé system can be extended to a Hamiltonian system on E_J. We call the system the J-th extended Painlevé system. We can verify moreover that each Hamiltonian $H_J(*)$ on the chart $V_J(*)$ is a polynomial of $x(*)$ and $y(*)$ of which the coefficeints are rational functions of t holormophic in B_J, where $(x(*), y(*), t)$ is the coordinates of $V_J(*) \simeq \mathbf{C}^2 \times B_J$. Therefore, the J-th extended Painlevé system defines a nonsingular foliation $\mathcal{F} = \mathcal{F}^J$ of E_J every leaf of which is transversal to fibers, and the well-known *Painlevé property* guarantees the *uniformity* of the foliation, that is,

> *for any point $P_0 \in E_J$, every curve in B_J with starting point $\pi_J(P_0)$ can be lifted on the leaf $\mathcal{F}_{P_0}$ passing through the point P_0.*

We remark that Hamiltonian system holomorphic on E_J and meromorphic on $\overline{E}_J := \bigsqcup_{t \in B_J} \overline{E}_J(t)$ is necessarily the J-th extended Painlevé system (Ref. 9, 11). Then we can roughly say that the manifold E_J knows everything about the J-th Painlevé system. On the other hand, Bäcklund transformations found by Okamoto (Ref. 6) can be easily derived from a description of the manifold, which is a work of H. Watanabe (Ref. 12). Anyway, it seems that the manifold E_J plays an important role in the study of the J-th Painlevé system.

Now we notice that the manifold E_J is obtained without a knowledge of Painlevé property. Therefore it is preferable to show uniformity of the foliation of E_J defined by the J-th extended Hamiltonian system by using only the description of E_J and the Hamiltonians $H_J(*)$ on $V_J(*)$. I have not yet obtained any purely geometric proof of uniformity, but the original proof of P. Painlevé (Ref. 7, 8) and M. Hukuhara (Ref. 1) can be made clear geometrically to some extent.

In the following, we give a proof of uniformity of the foliation associated with the IV-th extended Painlevé system as an example. The other cases are quite similar to the case $J = IV$. The most important part is how to choose auxiliary functions U_J on E_J which are a kind of Lyapunov functions. Although P. Painlevé did not explain why such functions were chosen, we can now understand the reason because we have the defining manifolds and the extended Painlevé systems on them. As we see in the next section, there are many choices of such functions, but the following ones seem to be simple and convenient to deal with:

$$\begin{aligned}
U_{VI}(x,y,t) &= H_{VI}(x,y,t) - (t-c)x(x-1)y/[t(t-1)(x-c)],\\
U_{V}(x,y,t) &= H_{V}(x,y,t) - (1-c)x(x-1)y/[t(x-c)],\\
U_{IV}(x,y,t) &= H_{IV}(x,y,t) + 2xy/(x-c),\\
U_{III'}(x,y,t) &= H_{III'}(x,y,t) + cxy/[t(x-c)],\\
U_{III}(x,y,t) &= H_{III}(x,y,t) + (tx+c)xy/[t(tx-c)],\\
U_{II}(x,y,t) &= H_{II}(x,y,t) + y/2x,\\
U_{I}(x,y,t) &= H_{I}(x,y,t) + y/2x.
\end{aligned}$$

Here, $U_J(x,y,t)$ is a restriction of U_J on the original chart $V_J(00) \simeq \mathbf{C}^2 \times B_J \ni (x,y,t)$, $H_J(x,y,t)$ is the Hamiltonian of the J-th Painlevé system, and the III'-th system is an system equivalent to the III-th one (Ref. 2).

2 The IV-th extended Painlevé system

The Hamiltonian $H := H_{IV}$ of the IV-th Painlevé system is given by

$$H(x,y,t) = 2xy^2 - \{x^2 + 2tx + 2\kappa_0\}y + \kappa_\infty x. \tag{2.1}$$

The manifold $E := E_{IV}$ which is the total space of $(E_{IV}, \pi_{IV}, B_{IV})$ is described as a patching of 4 copies $V(*) = \mathbf{C}^2 \times B \ni (x(*), y(*), t)$ $(B := B_{IV} = \mathbf{C},\ * = 00, 0\infty, \infty 0, \infty\infty)$ identified by the following rational and symplectic transformations:

$$x(00) = y(0\infty)\{\kappa_0 - x(0\infty)y(0\infty)\}, \quad y(00) = \frac{1}{y(0\infty)},$$

$$x(00) = \frac{1}{x(\infty 0)}, \quad y(00) = x(\infty 0)\{\kappa_\infty - x(\infty 0)y(\infty 0)\},$$

$$x(00) = \frac{1}{x(\infty\infty)},$$

$$y(00) = \frac{1}{2x(\infty\infty)} + t + x(\infty\infty)\{(\kappa_0 - \kappa_\infty - 1) - x(\infty\infty)y(\infty\infty)\}.$$

Here,

$$V(00) = \mathbf{C}^2 \times B \ni (x(00), y(00), t) = (x,y,t)$$

is the original chart of the IV-th Painlevé system. Note that each fiber $E(t)$ is a disjoint union of $V(00) \cap \pi^{-1}(t) = \mathbf{C}^2 \times t$ and 3 complex lines in $V(0\infty) \cap \pi^{-1}(t)$, $V(\infty 0) \cap \pi^{-1}(t)$, $V(\infty\infty) \cap \pi^{-1}(t)$:

$$E(t) = \mathbf{C}^2 \sqcup \{y(0\infty) = 0\} \sqcup \{x(\infty 0) = 0\} \sqcup \{x(\infty\infty) = 0\}.$$

Notice also the fiber is not compact.

The IV-th extended Painlevé system is given by 4 Hamiltonians $H(*) = H(*)(x(*), y(*), t)$ defined on $V(*)$. Notice that $\{H(*)\}_*$ does not define a function on E but they are related by

$$H(00) = H(0\infty) = H(\infty 0) = H(\infty\infty) - 1/x(\infty\infty),$$

because

$$dy(00) \wedge dx(00) - dH(00) \wedge dt = dy(*) \wedge dx(*) - dH(*) \wedge dt.$$

We can verify that each $H(*)$ is a polynomial of $x(*), y(*)$, and t. Therefore, the extension of $H = H(00)$ to the whole manifold E is meromorphic whose poles are only on $\{x(\infty\infty) = 0\}$ in $V(\infty\infty)$.

3 Lemmas

In this section, we give lemmas which are used in the proof of uniformity of the foliation associated with the IV-th Painlevé system. Note that curves in this note are always supposed to be of finite length and they are parametrized by their arc length.

Lemma 1(Painlevé). *Let $P_0 \in E, \gamma : [0, l] \to B, \gamma(0) = t_0 = \pi(P_0), \gamma(l) = a$ such that $\gamma - \{a\} := \gamma([0, l))$ can be lifted on the leaf $\mathcal{F}_{P_0}$ passing through the point P_0. We denote by $P(t)$ the point on the leaf $\mathcal{F}_{P_0}$ correspoinding to $t \in \gamma - \{a\}$. If there exists a sequence of points $\{t_n\}$ on $\gamma - \{a\}$ such that $t_n \to a$ and $P(t_n)$ tends to a point $P_\infty \in E(a)$ in E as $n \to \infty$, then the whole curve γ can be lifted on $\mathcal{F}_{P_0}$.*

This lemma is an immediate consequence of existence and uniqueness theorem for ordinary differential equations in the complex domains.

Lemma 2. *Let $P_0 \in E$ and let γ be a curve: $[0, l] \to B, \gamma(0) = t_0 = \pi(P_0), \gamma(l) = a$ such that $\gamma(s)$ is right differentiable at every $s \in [0, l)$ and $\gamma - \{a\}$ can be lifted on the leaf $\mathcal{F}_{P_0}$. If, for some $c \in \mathbf{C} - \{0\}, \rho > 0, r > 0$, it holds*

$$P(t) \in E(t) - \{(x, y, t) \in V(00) \,|\, |x - c| < \rho\} \qquad (3.1)$$

for every $t \in (\gamma - \{a\}) \cap D(a; r)$*, then the whole* γ *can be lifted on the leaf* $\mathcal{F}_{P_0}$*. Here,* $D(a; r) := \{t \in \mathbf{C} \mid |t - a| < r\}$*.*

In this note, we write the above (3.1) simply as

$$|x(t) - c| \geq \rho, \qquad t \in (\gamma - \{a\}) \cap D(a; r). \tag{3.2}$$

We give a proof of the lemma in Section 5.

Lemma 3. *Let* $f = f(x, y, t)$ *and* $g = g(x, y, t)$ *be functions which are holomorphic on* $\overline{G}(\rho_0) := \{(x, y, t) \in \mathbf{C}^3 \mid |x - c|, |1/y|, |t - a| \leq \rho_0\}$ *and satisfy, for some positive constants* l_{-1}, l_1, l, m *with* $7/8 < l_{-1}/l_1 < 1$*,*

$$l_{-1}|y| \leq |f(x, y, t)| \leq l_1|y|, \tag{3.3}$$

$$|g(x, y, t)| \leq m|y|^2, \tag{3.4}$$

$$|f(x, y, t) - f(x', y', t')| \leq l\{|y - y'| + (|x - x'| + |t - t'|)|y'|\} \tag{3.5}$$

for any $(x, y, t), (x', y', t') \in \overline{G}(\rho_0)$*. Then, for any* $l_2 (> l_1)$ *with* $l_{-1}/l_2 > 7/8$*, there exist some positive constants* ρ *and* ϵ *with* $\rho \leq \epsilon/(8m)$ *such that, for any* $(x_1, y_1, t_1) \in \overline{G}(\rho)$*, the solution* $(x(t), y(t))$ *of*

$$dx/dt = f(x, y, t), \quad dy/dt = g(x, y, t)$$

with $(x(t_1), y(t_1)) = (x_1, y_1)$ *exists on* $|t - t_1| \leq \epsilon/(l_2 m|y_1|)$ *and it satisfies*

$$|x(t) - c| \geq \frac{1}{4}\frac{\epsilon}{m}, \qquad \left(\frac{1}{2}\frac{\epsilon}{l_2 m|y_1|} \leq |t - t_1| \leq \frac{\epsilon}{l_2 m|y_1|}\right),$$

$$|x(t) - c| \geq \frac{3}{4}\frac{\epsilon}{m}, \qquad \left(|t - t_1| = \frac{\epsilon}{l_2 m|y_1|}\right),$$

$$|x(t) - c| \leq \frac{5}{8}\frac{\epsilon}{m}, \qquad \left(|t - t_1| \leq \frac{1}{2}\frac{\epsilon}{l_2 m|y_1|}\right).$$

This lemma (which is a system version of that in Ref. 1) says that, although $|x(t_1) - c|$ is small, $|x(t) - c|$ is bounded below if t parts suitably from t_1, because $|y(t_1)|$ is large. In this note, we omit its proof.

4 Sketch of the proof of uniformity

4.1 First half

Let us take a point $P_0 \in E$ arbitrarily and fix it. Denote by t_0 the point $\pi(P_0)$. Let S be a set of all positive r's such that any curve in

$$D(t_0; r) := \{t \in B = \mathbf{C} \mid |t - t_0| < r\}$$

(with starting point t_0) can be lifted on the leaf $\mathcal{F}_{P_0}$. If S is unbounded, then we have nothing to prove. Therefore, assuming that S is bounded, we derive a contradiction.

Suppose that S is bounded and let

$$R := \sup S.$$

Then there exists a point $a \in \partial D(t_0; R)$ such that no curve $\gamma : [0, l] \to B = \mathbf{C}$ with

$$\gamma(0) = t_0, \quad \gamma(l) = a, \quad \gamma([0, l)) \subset D(t_0; R)$$

can be lifted on the leaf $\mathcal{F}_{P_0}$. Here note that $\gamma - \{a\}$ can be lifted on $\mathcal{F}_{P_0}$ by the definition of R.

If there exist some $c \in \mathbf{C} - \{0\}, \rho > 0, r > 0$ such that (3.2) holds on the line segment $\overline{t_0 a} - \{a\}$, then, by Lemma 2, the whole line segment $\overline{t_0 a}$ can be lifted on $\mathcal{F}_{P_0}$. Therefore, for any $c \in \mathbf{C} - \{0\}, \rho > 0, r > 0$, the inequality (3.2) does not hold, namely, for any $c \in \mathbf{C} - \{0\}, \rho > 0, r > 0$, there exists a $t \in (\overline{t_0 a} - \{a\}) \cap D(a; r)$ such that

$$|x(t) - c| \le \rho.$$

If, for any $c \in \mathbf{C} - \{0\}, \rho > 0, r > 0$, there exists a $t \in (\overline{t_0 a} - \{a\}) \cap D(a; r)$ such that

$$|x(t) - c| \le \rho, \quad |1/y(t)| > \rho,$$

then, for fixed $c \in \mathbf{C} - \{0\}$ and $\rho > 0$, we can find a sequence $\{t_n\} \subset \overline{t_0 a} - \{a\}$ such that $t_n \to a$ and $(x(t_n), y(t_n), t_n) \in V(00) \subset E$ tends to a point $(x_\infty, y_\infty, a) \in V(00) \subset E$ in E as $n \to \infty$, and then $\overline{t_0 a}$ can be lifted on the leaf $\mathcal{F}_{P_0}$ by Lemma 1. Therefore, for any $c \in \mathbf{C} - \{0\}, \rho > 0, r > 0$, there exists a $t \in (\overline{t_0 a} - \{a\}) \cap D(a; r)$ such that

$$|x(t) - c| \le \rho, \quad |1/y(t)| \le \rho,$$

namely, for any $c \in \mathbf{C} - \{0\}, \rho > 0$, the point a is an accumulation point of a set

$$\{t \in \overline{t_0 a} - \{a\} \mid |x(t) - c|, |1/y(t)| \le \rho\}. \tag{4.1}$$

Take $c \in \mathbf{C} - \{0\}$ arbitrarily and fix it. By the form of the Hamiltonian $H = H(00)$ and the fact $c \neq 0$, we can take constants $0 < \rho_0 << 1, l_{-1}, l_1, l, m$ with $7/8 < l_{-1}/l_1 < 1$ so that $f := \partial H/\partial y$ and $g := -\partial H/\partial x$ satisfy the inequalities (3.3), (3.4), (3.5). Then, for arbitrarily chosen $l_2(> l_1)$ with $l_{-1}/l_2 > 7/8$, we can take some $\rho, \epsilon > 0$ with $\rho \leq \epsilon/(8m)$ so that the conclusion of Lemma 3 holds.

4.2 Second half

Let us fix $c \neq 0, \rho > 0$ and other constants as above. Notice that the point $t = a$ is an accumulation point of the set (4.1).

We fist take a point $t_1 \in \overline{t_0 a} - \{a\}$ such that

$$|x(t_1) - c|, |1/y(t_1)| \leq \rho.$$

Let τ_1', τ_1'' (or t_1', t_1'') be the intersection points of the line $\overline{t_0 a}$ and the circle $\{|t - t_1| = \epsilon/(2l_2 m|y_1|)\}$ (or $\{|t - t_1| = \epsilon/(l_2 m|y_1|)\}$) such that τ_1'' (or t_1'') is nearer to a than τ_1' (or t_1'), where $y_1 := y(t_1)$. We replace line segment $\overline{\tau_1' \tau_1''}$ by a semi-circle C_1 from τ_1' to τ_1'' on the circle $\{|t - t_1| = \epsilon/(2l_2 m|y_1|)\}$. Then $|x(t) - c| \geq \epsilon/(4m)(> \rho)$ on C_1, $\overline{t_1' \tau_1'}$, and $\overline{\tau_1'' t_1''}$. We see that $\overline{D}(t_1; \epsilon/(l_2 m|y_1|)) \subset D(t_0; R)$ by our assumption that S is bounded.

Next we take a point t_2 nearest to t_1'' in

$$\{t \in \overline{t_1'' a} - \{a\} \mid |x(t) - c|, |y(t)| \leq \rho\}.$$

Let τ_2', τ_2'' (or t_2', t_2'') be the intersection points of $\overline{t_0 a}$ and $\{|t - t_2| = \epsilon/(2l_2 m|y_2|)\}$ (or $\{|t - t_2| = \epsilon/(l_2 m|y_2|)\}$) such that τ_2'' (or t_2'') is nearer to a than τ_2' (or t_2'), where $y_2 := y(t_2)$. We replace $\overline{\tau_2' \tau_2''}$ by a semi-circle C_2 from τ_2' to τ_2'' on the circle $\{|t - t_2| = \epsilon/(2l_2 m|y_2|)\}$. Then $|x(t) - c| \geq \epsilon/(4m)(> \rho)$ on C_2, $\overline{t_2' \tau_2'}$, and $\overline{\tau_2'' t_2''}$. Notice that $\overline{D}(t_2; \epsilon/(l_2 m|y_2|)) \subset D(t_0; R)$.

The two semi-circles C_1 and C_2 are separated, becuase

$$\overline{D}(t_1; \epsilon/(l_2 m|y_1|)) \cap \overline{D}(t_2; \epsilon/(2l_2 m|y_2|)) = \phi,$$

which is derived as: if the left-hand side is not empty, t_1'' is a point in it, which yields $|x(t_1'') - c| \geq (3\epsilon)/(4m)$ and $\leq (5\epsilon)/(8m)$.

By the assumption that S is bounded, we can continue the above processes infinitely many times, namely we can choose infinitely many semi-circles $C_n, n = 1, 2, \ldots$ (of which the centers are $t_n, n = 1, 2, \ldots$) in $D(t_0; R)$ which are separated mutually. We can also verify that $t_n \to a$ as $n \to \infty$ and $|x(t) - c| > \rho,\ t \in \overline{t_k'' \tau_{k+1}'}$ for all sufficiently large k.

Let γ be a curve obtained from $\overline{t_0 a}$ by replacement of subsegments by $\{C_n\}$. Then $|x(t)-c|>\rho,\ t\in(\gamma-\{a\})\cap D(a;r)$ for some positive small r, which implies that γ can be lifted on $\mathcal{F}_{P_0}$. This contradicts our assumption.

5 Proof of Lemma 2

5.1 Auxiliary function U

We introduce an auxiliary function U on E by

$$U|_{V(00)} = H(x,y,t) + \frac{2xy}{x-c}, \tag{5.1}$$

where $U|_{V(00)}$ is the restriction of U on $V(00)$, $(x,y,t)=(x(00),y(00),t)$ is the coordinate system of the the original chart $V(00)$, and $H(x,y,t)$ is the Hamiltonian (2.1) of the IV-the Painlevé system. We write (5.1) simply as

$$U = 2xy^2 - \left\{x^2 + 2tx + 2\kappa_0 - \frac{2x}{x-c}\right\} y + \kappa_\infty x. \tag{5.2}$$

The function U is meromorphic on E having poles on $\{x=c\}$ in $V(00)$. As we have noticed in Section 2, the extension of the original Hamiltonian $H=H(00)$ to the whole manifold E has poles on $\{x(\infty\infty)=0\}$ in $V(\infty\infty)$. The term $2xy/(x-c)$ is added in order to move the pole divisor from $\{x(\infty\infty)=0\}$ to $\{x=c\}$.

5.2 Boundedness of $U(P(t))$ on $\gamma-\{a\}$

Let U' be a function on E such that

$$dU(P(t))/dt = U'(P(t))$$

for any local leaf $\bigsqcup_t P(t)$ of the foliation. Then the restriction of U' on $V(00)$ is given by

$$U'|_{V(00)} = \frac{\partial H}{\partial t} - \frac{2cy}{(x-c)^2}\frac{\partial H}{\partial y} - \frac{2x}{x-c}\frac{\partial H}{\partial x}. \tag{5.3}$$

By eliminating the variable y in (5.2) and (5.3), we obtain a relation

$$(U'+AU+B)^2 + C(U'+AU+B) + (DU+E) = 0$$

where A, B, C, D and E are certain rational functions of x and t. We can verify that these functions are holomorphic in $(\mathbf{P} - \{c\}) \times \mathbf{C} \ni (x, t)$, which implies

$$|U'| < m|U| + n \tag{5.4}$$

on $\bigsqcup_{t \in \gamma}(E(t) - \{(x, y, t) \in V(00) \mid |x - c| < \rho\})$, for some positive m and n.

We set

$$U(s) := U(P(t(s))), \qquad s \in [0, l).$$

Since $U(s)$ is right differentiable at any $s \in [0, l)$, $|U(s)|$ is also right differentiable at $s \in [0, l)$ and from the inequality (5.4), it follows

$$D^+|U(s)| < m|U(s)| + n, \qquad s \in [0, l).$$

Hence $U(P(t))$ is bounded on $\gamma - \{a\}$.

5.3 Reduction to Lemma 1

In this part, the description of the manifold E_J seems to play the most important role. Since $U(P(t))$ is bounded on $\gamma - \{a\}$, we can choose a sequence of points $\{t_n\}$ on $\gamma - \{a\}$ so that $U(P(t_n))$ tends to a constant $U_\infty \in \mathbf{C}$ and $x(t_n)$ tends to a $x_\infty \in \mathbf{P} - \{c\}$ as $n \to \infty$. Now we notice the relation among $y(t), x(t)$ and $U(t) := U(P(t))$ for $t \in \gamma - \{a\}$:

$$y = \frac{\{x^2 + 2tx + 2\kappa_0 - 2x/(x - c)\} \pm \sqrt{\{\ldots\}^2 - 8x(\kappa_\infty x - U)}}{4x}. \tag{5.5}$$

The case $x_\infty \neq 0, \infty$. By (5.5), $\{y(t_n)\}_n$ is bounded. Then we take a subsequence of $\{t_n\}$ which is also denoted by $\{t_n\}$ so that $(x(t_n), y(t_n), t_n) \in V(00)$ tends to a point $(x_\infty, y_\infty, a) \in V(00)$. Hence, from Lemma 1, $\gamma - \{a\}$ is lifted on the leaf $\mathcal{F}_{P_0}$.

The case $x_\infty = 0$. If $\{y(t_n)\}_n$ is bounded, then $\gamma - \{a\}$ is lifted on $\mathcal{F}_{P_0}$ by Lemma 1, and hence we consider the case where $\{y(t_n)\}_n$ is unbouded. We take a subsequence which is denoted by the same symbol so that $x(t_n) \to 0$ and $y(t_n) \to \infty$ as $n \to \infty$. In this case, we observe the behavior of $\{P(t_n)\}_n$ by the use of chart $V(0\infty)$. Let $(X, Y) := (x(0\infty), y(0\infty))$, then

$$X = y(\kappa_0 - xy), \qquad Y = 1/y$$

and $Y(t_n) \to 0$ as $n \to \infty$. Since

$$xy = \frac{1}{2}\left\{x^2 + 2tx + 2\kappa_0 - \frac{2x}{x - c}\right\} - \frac{1}{2y}(\kappa_\infty x - U)$$

we see

$$x(t_n)y(t_n) \to \kappa_0$$

as $n \to \infty$. From this and from

$$X = y(\kappa_0 - xy) = -\frac{1}{2}x \cdot xy - t \cdot xy + \frac{xy}{x-c} + \frac{\kappa_\infty x - U}{2}$$

if follows that

$$X(t_n) \to -a\kappa_0 - \kappa_0/(a-c) - U_\infty/2 =: X_\infty \in \mathbf{C}$$

namely, $(X(t_n), Y(t_n), t_n) \to (X_\infty, 0, a) \in V(0\infty)$. Hence $\gamma - \{a\}$ is lifted on $\mathcal{F}_{P_0}$.

The case $x_\infty = \infty$. Since $|x(t_n)|$ is large for every large n, we have

$$y(t_n) = x(t_n)/2 + t_n + O(1/x(t_n)) \quad \text{or} \quad O(1/x(t_n))$$

by (5.5). Therefore, we can take a subsequence so that $y(t_n) \to \infty$ or $y(t_n) \to 0$ as $n \to \infty$.

The case $x(t_n) \to \infty, y(t_n) \to \infty$. In this case, we observe $\{P(t_n)\}_n$ by the use of $V(\infty\infty)$. Let $(X, Y) := (x(\infty\infty), y(\infty\infty))$, then

$$X = 1/x, \quad Y = x\left\{(\kappa_0 - \kappa_\infty - 1) - x(y - x/2 - t)\right\}.$$

We first obtain $x(t_n)/y(t_n) \to 2$ and next

$$(X(t_n), Y(t_n), t_n) \to (0, c - 2\kappa_\infty a - U_\infty, a) \in V(\infty\infty)$$

by the use of

$$Y = c\frac{x}{x-c} - \kappa_\infty t\frac{x}{y} - \kappa_0\kappa_\infty\frac{1}{y} + \kappa_\infty\frac{x}{x-c}\frac{1}{y} + \frac{{\kappa_\infty}^2}{2}\frac{x}{y}\frac{1}{y} - \frac{\kappa_\infty}{2}\frac{U}{y^2} - \frac{U}{2}\frac{x}{y}.$$

Hence $\gamma - \{a\}$ is lifted on $\mathcal{F}_{P_0}$.

The case $x(t_n) \to \infty, y(t_n) \to 0$. In this case, we use $V(\infty 0)$. Let $(X, Y) := (x(\infty 0), y(\infty 0))$, then $X = 1/x, Y = x(\kappa_\infty - xy)$. We have first $x(t_n)y(t_n) \to \kappa_\infty$ and next

$$(X(t_n), Y(t_n), t_n) \to (0, 2\kappa_\infty a - 2\kappa_\infty/(a-c) + U_\infty, a) \in V(\infty 0).$$

Hence $\gamma - \{a\}$ is lifted on $\mathcal{F}_{P_0}$.

We have thus completed the proof of Lemma 2.

References

1. Hukuhara, M., I. Meromorfeco de la solvo de (F) $y'' = 6y^2 + x$, II. Diferenciala sistemo de Schlesinger, III. Ekvivalenteco de la diferenciala sistemo de Schlesinger kaj tiu de Garnier, IV. La ekvacio (VI) de Painlevé-Gambier, unpublished lecture note.

2. Iwasaki, K., Kimura, H., Shimomura, S. and Yoshida, M., *From Gauss to Painlevé*, 1991, Vieweg.

3. Joshi, N. and Kruskal, M.D., A direct proof that solutions of six Painlevé euqtions have no movable singularities exept poles, Stud. Appl. Math., **93**(1994), 187-207.

4. Miwa, T., Painlevé property of monodromy preserving deformation equations and the analyticity of τ functions, Publ. RIMS, Kyoto Univ., **17**(1981), 703-721.

5. Okamoto, K., Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Japan. J. Math., **5**(1979), 1-79.

6. Okamoto, K., Studies on the Painlevé equations, I, Ann. Mat. Pura Appl., **146**(1987), 337-381; II, Jap. J. Math., **13**(1987), 47-76; III, Math. Ann., **275**(1986), 221-256; IV, Funkcial. Ekvac., **30** (1987), 305-332.

7. Painalvé, P., Mémoire sur les équations différentielles dont l'integrale generale est uniforme, Bull. Soc. Math. France, **28**(1900), 206-261.

8. Painlevé, P., Sur les équations différentielles du second ordre à points critiques fixes, C. R. Acad. Sci. Paris, **143**(1906), 1111-1117.

9. Shioda, T. and Takano, K., On some Hamiltonian structures of Painlevé systems I, Funkcial. Ekvac., **40**(1997), 271-291.

10. Matano, T., Matumiya, A. and Takano, K., On some Hamiltonian structures of Painlevé systems II, to appear.

11. Matumiya, A., On some Hamiltonian structures of Painlevé systems III, Kumamoto J. Math., **10**(1997), 45-73.

12. Watanabe, H., Birational canonical transformations and classical solutions of the sixth Painlevé equation, preprint.