Discrete constant mean curvature surfaces via conserved quantities in any space form

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Our purpose here is to present a definition for discrete constant mean curvature (CMC) h surfaces[1] in any of the three space forms Euclidean 3-space R^3, spherical 3-space S^3 and hyperbolic 3-space H^3. This new definition is equivalent to the previously known definitions [2] in the case of R^3. It also satisfies a Calapso transformation relation (the Lawson correspondence), suggesting the definition is also natural for the space form S^3, and for CMC surfaces with $h \geq 1$ in H^3. The definition is the first one for CMC surfaces with $-1 < h < 1$ in H^3.

To motivate this definition for discrete CMC surfaces, we first consider the case of smooth surfaces, and we begin by describing the 3-dimensional space forms using the 5-dimensional Minkowski space $R^{4,1}$.

Minkowski 5-space. We give a 2×2 matrix formulation for Minkowski 5-space. Let H denote the quaternions and $\operatorname{Im} H$ the imaginary quaternions.

$$\mathbb{R}^{4,1} = \left\{ X = \begin{pmatrix} x & x_\infty \\ x_0 & -x \end{pmatrix} \,\middle|\, x \in \operatorname{Im} H, x_0, x_\infty \in \mathbb{R} \right\}$$

with signature $(+,+,+,+,-)$ Minkowski metric $\langle X, Y\rangle$ such that $\langle X, Y\rangle \cdot I = -\frac{1}{2}(XY + YX)$, I = identity matrix. The 4-dimensional light cone is $\mathbb{L}^4 = \{X \in \mathbb{R}^{4,1} \,|\, ||X||^2 = 0\}$. We can make the 3-dimensional space forms as follows: A space form M is $M = \mathbb{L}^4 \cap \{X \,|\, \langle X, Q\rangle = -I\}$ for any nonzero $Q \in \mathbb{R}^{4,1}$. It turns out that M has curvature κ, where $Q^2 = \kappa \cdot I$, so without loss of generality we can obtain any space form by choosing

$$Q = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix}, \quad \text{and then} \quad M = \left\{ X = \frac{2}{1 - \kappa x^2} \cdot \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} \right\},$$

which is equivalent to $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \cup \{\infty\} \,|\, x_1^2 + x_2^2 + x_3^2 \neq -\kappa^{-1}\}$, where $x = x_1 i + x_2 j + x_3 k \in \operatorname{Im} H$. Note that when $\kappa < 0$, M becomes two copies of $H^3(\kappa)$.

Smooth surfaces in space forms. Let

$$x = x_1(u, v)i + x_2(u, v)j + x_3(u, v)k \approx X \in M$$

be a surface in M. Assume (u, v) is a conformal curvature-line coordinate system (every CMC surface can be parametrized this way). First we define the Christoffel transformation x^*, which for a CMC surface in R^3 gives the parallel CMC surface:

Definition. *The Christoffel transformation of x is any x^* (defined in R^3 up to translation) such that $dx^* = x_u^{-1} du - x_v^{-1} dv$.*

In the next definition, the nonzero real constant c can be chosen freely:

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Definition. *For some* $c \in R\backslash\{0\}\}$, *we set* $\tau = c\begin{pmatrix} xdx^* & -xdx^*x \\ dx^* & -dx^*x \end{pmatrix}$. *If there exist smooth* Q *and* Z *in* $R^{4,1}$ *depending on* (u,v) *such that*

$$d(Q+\lambda Z) = (Q+\lambda Z)\lambda\tau - \lambda\tau(Q+\lambda Z) \tag{1}$$

holds for all $\lambda \in R$, *then we call* $Q+\lambda Z$ *a* linear conserved quantity *of* x.

Some properties of linear conserved quantities are immediate: Q and Z^2 are constant; $F\tau = \tau F = 0$; $F \perp Z$ and $F \perp dZ$. Properties like this can be utilized to prove the following theorem:

Theorem. [1] *The surface* x *is constant mean curvature in a space form* M *(produced by* $Q \neq 0$*) if and only if there exists (for that* Q*) a linear conserved quantity* $Q+\lambda Z$.

Isothermic discrete surfaces and their Christoffel transforms. Consider a discrete surface $\mathfrak{f}_p \in \operatorname{Im} H$, where p is any point in a discrete lattice domain. Consider one quadrilateral in the lattice with vertices p, q, r, s ordered counterclockwise about the quadrilateral. We define the cross ratio of this quadrilateral as

$$q_{pqrs} = (\mathfrak{f}_q - \mathfrak{f}_p)(\mathfrak{f}_r - \mathfrak{f}_q)^{-1}(\mathfrak{f}_s - \mathfrak{f}_r)(\mathfrak{f}_p - \mathfrak{f}_s)^{-1} \; .$$

When, for every quadrilateral, we can write the cross ratio as

$$q_{pqrs} = a_{pq}/a_{ps} \in \mathbb{R}$$

so that the function a_{pq} defined on the edges of $\mathfrak{f}$ satisfies

$$a_{pq} = a_{sr} \in \mathbb{R} \;\text{ and }\; a_{ps} = a_{qr} \in \mathbb{R} \; ,$$

then we say that $\mathfrak{f}$ is *isothermic*.

We can define the Christoffel transform $\mathfrak{f}^*$ of $\mathfrak{f}$ by

$$d\mathfrak{f}^*_{pq} d\mathfrak{f}_{pq} = a_{pq} \; .$$

We can then prove the following:

Lemma. [2] *If* $\mathfrak{f}$ *is isothermic, then there exists a discrete surface* $\mathfrak{f}^*$ *satisfying the above equation for* $d\mathfrak{f}^*$, *and* $\mathfrak{f}^*$ *is isothermic with the same cross ratios as* $\mathfrak{f}$.

Linear conserved quantities. We can now discretize (1) to obtain

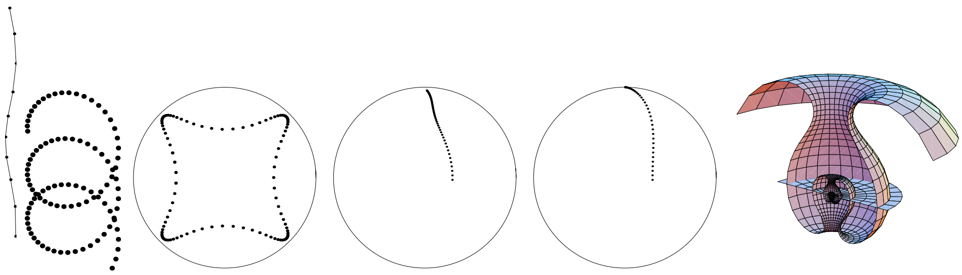
$$(1+\lambda\tau_{pq})(Q+\lambda Z)_q = (Q+\lambda Z)_p(1+\lambda\tau_{pq}) \; , \tag{2}$$

where $\lambda \in \mathbb{R}$ and $Q, Z \in R^{4,1}$ are functions on the lattice domain, and

$$\tau_{pq} = \begin{pmatrix} \mathfrak{f}_p d\mathfrak{f}^*_{pq} & -\mathfrak{f}_p d\mathfrak{f}^*_{pq}\mathfrak{f}_q \\ d\mathfrak{f}^*_{pq} & -d\mathfrak{f}^*_{pq}\mathfrak{f}_q \end{pmatrix} \; .$$

We now come to the goal of this talk:

Definition. *If a linear conserved quantity* $Q+\lambda Z$, $Q \neq 0$, *exists for an isothermic discrete surface* $\mathfrak{f}$, *we say that* $\mathfrak{f}$ *is of constant mean curvature in the space form* M *determined by* Q.



Equation (2) can be extended to define polynomial conserved quantities.

In the figure, we show discrete CMC surfaces of revolution. The first two curves are profile curves for discrete nonminimal CMC surfaces of revolution in R^3, the first being unduloidal and the second nodoidal. (For each of these two curves, the axis of rotation producing the surface is a vertical line drawn to the left of the curve, and is not shown in the figure.) The third picture shows the profile curve for a discrete CMC surface of revolution in S^3, where S^3 is stereographically projected to R^3, and the shown circle is a geodesic of S^3 that is also the axis of the surface – and furthermore, this example has a periodicity that causes it to close on itself and form a torus. The final three pictures show discrete CMC surfaces of revolution in H^3. The first two, with $H > 1$ and $H = 1$ respectively, are shown in the Poincare model, and the first is unduloidal while the second looks similar to a smooth embedded catenoid cousin. (For these two curves, the corresponding axis of revolution is the vertical line between the uppermost and lowermost points of the circle shown, and this circle lies in the boundary sphere at infinity of H^3.)

The last picture is a minimal surface that lies in both copies of $M = H^3$, and the horizontal plane shown here is the virtual boundary at infinity of two copies of the halfspace model for H^3. This example was not known before, because the notion of discrete CMC was not defined before in this case.

References

[1] F. Burstall, D. Calderbank, *Conformal submanifold geometry*, in preparation.

[2] A. Bobenko, U. Pinkall, *Discretization of surfaces and integrable systems*, Oxford Lect. Ser. Math. Appl. **16** (1999), 3–58.