Computers and Mathematics: Applications of computers to integrable systems methods for constructing constant mean curvature surfaces

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This article is about using computer algorithms and graphics in the study of constant mean curvature surfaces. The advantages of using computers in surface theory is clear, as it helps one to visually see the geometric behavior of the objects of study in a way that is not possible in most fields of mathematics. But on a more fundamental level it is about turning an infinite dimensional problem (that a computer can never truly solve) into a finite dimensional one (that is easily solved with a computer), an idea that applies equally well in a great variety of mathematical fields, and is in no way unique to surface theory.

Think of a simple first order smooth ordinary differential equation with a given initial condition. If you cannot write down its solution explicitly, you might think about finding a discrete approximate solution by using the Euler algorithm or Runga-Kutta algorithm, just to have some initial idea how the smooth solution behaves. In this case, your interest in the approximate solution is only as a stepping stone for understanding the smooth true solution. We can think of the equation (i.e. the algorithm) for the discrete approximate solution as a "finite dimensional" problem because the full space of objects (a vector space of discrete functions) that can be inserted to test for validity in the equation is finite dimensional. Likewise, we can call the smooth differential equation an "infinite dimensional" problem, because the objects insertable into the equation form an infinite dimensional vector space. This is a somewhat unconventional way to use the expressions "finite dimensional" and "infinite dimensional", but some geometers do use these expressions in this way in conversations, although generally not in papers they write.

Or you might instead look at a related ordinary difference equation, with little concern that the resulting discrete solution approximates the smooth solution, and rather be more concerned that the difference equation maintains some property found in the smooth differential equation that you deem important. In this case, as your primary interest is the "finite dimensional" difference equation situation itself, you might discard the smooth equation altogether, or you might acknowledge the existence of the smooth equation but regard it only as an incidental limiting case of the difference equation you care much more about.

Both approaches are of interest, but for clearly different reasons, and are philosophically quite separate, although both clearly benefit from the existence of computers. Both are now common in surface theory, though usually involving partial differential equations, not ordinary ones. Regarding the second approach, finding discrete analogs of smooth objects has recently become an important theme in mathematics, appearing in a variety of places in analysis and geometry. So it

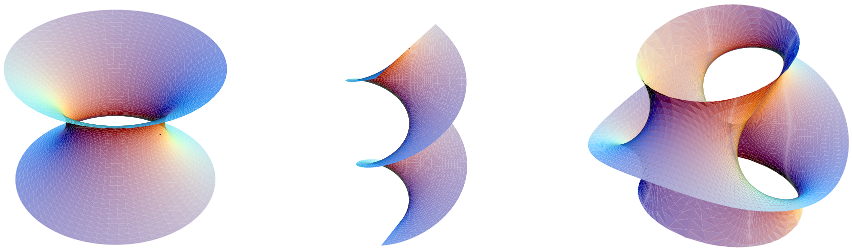

Figure 1: Three minimal surfaces, the catenoid, helicoid and Costa surface, in R^3.

is natural to search for discrete analogs of smooth minimal and constant mean curvature surfaces. But there is no single definitive way to define these analogs; the definition one chooses depends on which properties of smooth minimal and constant mean curvature surfaces one wishes to emulate in the discrete case.

The first approach is used by researchers in surface theory who want to understand smooth surfaces in cases where they have no other more elegant way to attack a problem. A good example of this is:

> **The Costa Surface:** A minimal surface M in Euclidean 3-space R^3 is a surface so that for every point $p \in M$, there exists a neighborhood $U \subset M$ of p such that U is the unique surface of least area with respect to its boundary ∂U. Soap films not containing bounded pockets of air minimize area with respect to their boundaries, and thus are modelled by minimal surfaces. (The equivalent definition usually used by geometers is that the principal curvatures have equal absolute value but opposite sign at every $p \in M$. But we choose the definition above, as it best explains why these surfaces are called "minimal".) The simplest example is a plane in R^3, and two other rather simple examples are: 1) the catenoid, a surface of revolution produced by revolving a catenary and parametrized by
>
> $$\{(\cosh u \cos v, \cosh u \sin v, u) \in R^3 \,|\, u \in R, v \in [0, 2\pi)\} ,$$
>
> where R denotes the real numbers, and 2) the helicoid, foliated by straight lines and parametrized by
>
> $$\{(\sinh u \cos v, \sinh u \sin v, v) \in R^3 \,|\, u, v \in R\} .$$
>
> The famous Weierstrass representation says that all minimal surfaces can be locally parametrized by pairs of meromorphic functions f, g defined on Riemann surfaces Σ with local complex coordinates z, by using path integrals:
>
> $$\text{Re} \int_{z_0}^{z} (1 - g^2, i + ig^2, 2g) f dz , \quad i = \sqrt{-1} .$$
>
> There had been a long standing conjecture that the only complete embedded minimal surfaces with finite topology in R^3 are the plane and catenoid and helicoid. Then in 1984, Costa [4] found a complete minimal surface homeomorphic to a torus minus three

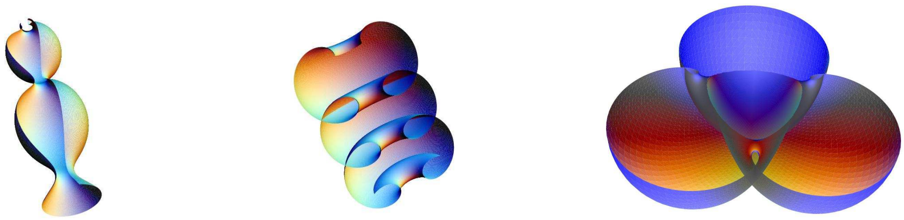

Figure 2: Cut-aways of three constant mean curvature surfaces, a Delaunay unduloid, a Delaunay nodoid and a Wente torus, in R^3. The first two are surfaces of revolution. (Graphics made by Kouichi Shimose.)

points that seemed it might actually be embedded, because it was at least embedded outside of a compact set in R^3. In 1985, Hoffman and Meeks [7] confirmed it is a counterexample by proving it is embedded, but only after numerics led them to see that the surface possessed certain lines and planes of symmetry that were useful for the proof. Though their final proof used no numerics, the numerics helped them to find it. They were using the first approach, and the "finite dimensional" approximation was only a tool that they discarded once it had enlightened them. It is not hard to see how the numerics can be made, by noting that the Costa surface has a Weierstrass representation as above with

$$\Sigma = \{(z, w) \in (C \cup \{\infty\})^2 \mid w^2 = z(z^2 - 1)\} \setminus \{(-1, 0), (1, 0), (\infty, \infty)\} ,$$

and

$$g = B/w \ , \quad f = w/(z^2 - 1) \ ,$$

where B is the constant

$$B = \sqrt{2 \int_0^1 \left(\frac{t}{1 - t^2} \right)^{1/2} dt \Bigg/ \int_0^1 \frac{dt}{t(1 - t^2)^{1/2}}} \ .$$

Examples of this first approach existed even well before the age of computers, as the next example shows:

The Wente tori: Constant mean curvature (CMC) surfaces can be defined just like minimal surfaces, except that now there is a volume constraint. A CMC surface M in R^3 is a surface so that for every point $p \in M$, there exists a neighborhood $U \subset M$ of p such that U is the unique surface of least area with respect to its boundary ∂U, amongst the set of all surfaces with the same oriented volume. We say that two bounded simply-connected surfaces U and $\hat{U}$ with $\partial U = \partial \hat{U}$ have the same oriented volume if one can deform U into $\hat{U}$ through a family of surfaces that never changes the boundary and never changes the neighboring volume to each side of the surfaces. Soap films that do contain bounded pockets of air minimize area with respect to their boundaries and fixed volumes, and thus are modelled by CMC surfaces. (The geometers' equivalent definition is that the average of the principal curvatures, i.e. the mean curvature, is constant along M. Now it is the geometers' definition that explains why these surfaces are called "constant mean curvature".) Simple examples are the round sphere and round cylinder. Less trivial examples are Delaunay surfaces of revolution, parametrizable explicitly in terms of the nonconstant periodic Jacobi elliptic function $v(x)$ satisfying

$$(v')^2 = -(v^2 - 4s^2)(v^2 - 4t^2) \ , \quad v(0) = 2|t| \ ,$$

with $s, t \in R \setminus \{0\}$, $s \neq t$ and $s + t = 1/2$, and the elliptic integral of the third kind

$$\int_0^x \frac{4st}{4st + v^2(\rho)} d\rho \ .$$

Hopf [9] asked whether any compact CMC surface without boundary in R^3 must be a round sphere. (He did not conjecture it, rather he only asked it.) He himself proved it is true when the surface is simply connected. Alexandrov proved it when the surface is embedded, using the maximum principle for second-order elliptic differential equations. However, Wente [15] showed it is false in general, by finding compact nonembedded CMC surfaces without boundary and of genus 1. These tori can be described in terms of Jacobi elliptic functions and integrals, like the Delaunay surfaces are.

But in fact Enneper's student Voretzsch was already studying local pieces of surfaces that include Wente tori well over 100 years ago. Tables of his data, obviously found without the aid of a computer, are listed in his thesis [14]. Voretzsch reduced the problem to a numerical study of elliptic functions by assuming the surface has a family of planar curvature lines, which is true of the Wente tori (although this was only later noticed by U. Abresch [1]). He used his data to make a plaster model, which later disappeared for unknown reasons. Recently a master's degree student at TU-Berlin, now using the Mathematica program, checked that Voretzsch's data was very accurate [16]. There are presently plans under way at TU-Berlin to reproduce Voretzsch's plaster model with a 3D printer.

If Voretzsch had thought to see if his surfaces could close into compact surfaces, Wente tori might have been known long before Hopf would have even asked his question.

To apply the second approach, on the other hand, one must decide what properties of smooth CMC surfaces one would like to see preserved in the discrete CMC surfaces, and different choices of those properties result in genuinely different theories.

One choice, following the definitions given above, is to demand that the discrete CMC surfaces also locally minimize area with respect to variations of the surface that preserve volume to each side. For this property, one would choose the discrete surfaces to be triangulated, i.e. as surfaces made by gluing triangles together along edges, and then consider variations that continuously move the vertices while preserving the simplicial structure. If any variation that moves just one interior vertex and that preserves the volume to each side of the surface will never decrease area, we can say that this discrete surface is of constant mean curvature. A proponent of this approach is K. Polthier [12], [13], and there exists a very user-friendly and versatile software for finding these discrete CMC surfaces numerically, the Surface Evolver by K. Brakke.

software	location
Surface Evolver	K. Brakke, www.susqu.edu/brakke
	Knoppix/Math, www.knoppix-math.org
JavaView	K. Polthier, www.javaview.de
CMCLab, Java version	Tokyo Metro. Univ., tmugs.math.metro-u.ac.jp
CMCLab, Linux version	N. Schmitt, www.gang.umass.edu
	Knoppix/Math, www.knoppix-math.org

Table 1: Recommended related freeware.

But another choice of the property to be preserved, now commonly used, is as follows: The governing equation, i.e. the Gauss equation, when using conformal coordinates, for the local existence of a CMC surface is the sinh-Gordon equation

$$\partial_{\bar{z}}\partial_z u + \sinh u = 0 \ ,$$

where z is a local complex coordinate on a Riemann surface.

To be honest, this is not quite true at umbilic points where the two principle curvatures are equal, because more generally the Gauss equation is

$$\partial_{\bar{z}}\partial_z u - \tfrac{1}{2}Q\bar{Q}e^{-u} + 2H^2 e^u = 0$$

for H the constant mean curvature and for some holomorphic function $Q = Q(z)$, and the umbilic points are where Q is zero. But these umbilic points are merely isolated on any CMC surface other than the round sphere, and away from those points a conformal coordinate change and homothety of the surface will transform this equation into the sinh-Gordon equation above. Let us ignore umbilic points in this introductory article, so then CMC surfaces correspond locally to solutions of the sinh-Gordon equation.

So we can now obtain discrete CMC surfaces via a discretization of that integrable system (the sinh-Gordon equation). In this case, the area-minimizing property has been lost, so there is no reason to think about variations of the surface, and so one can consider the surfaces as a mesh of planar quadrilaterals (for which one cannot freely move the vertices, as planarity will then be lost), rather than of triangles. Perhaps the first breakthrough in this field was a paper by Wunderlich [17]. The TU-Berlin geometry group, under the leadership of A. Bobenko and U. Pinkall, rediscovered Wunderlich's results and then developed the field further.

At first, preserving the area-minimizing property for the discrete surfaces might seem preferable to preserving a relationship with integrable systems, as the former property is fundamental to how the smooth surfaces are defined, while the latter property is something that one later discovers in their mathematical structure. Without doubt, the first way is important, but there are good reasons for considering the second way too. The second way, by preserving relations to integrable systems, preserves much of the interesting underlying mathematical structure [2], [6]. (For example, only the second way gives discrete versions of the Bianchi permutability theorem, and discrete transformations of Backlund, Darboux or Ribaucour type.)

It is interesting that one cannot have it both ways when discretizing, that is, one cannot simultaneously preserve the area-minimizing property and the relationships to integrable systems.

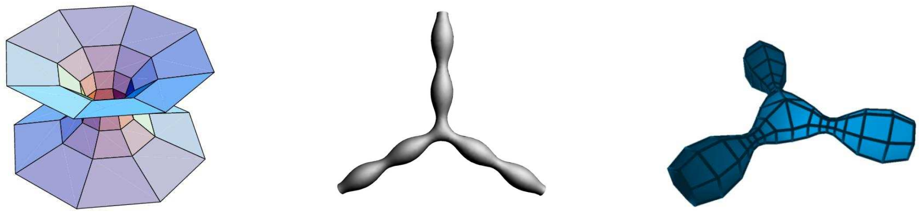

Figure 3: A discrete catenoid found in [3] made via an integrable systems viewpoint. (Discrete catenoids made via a variational viewpoint can easily be viewed using Polthier's JavaView software and would look similar, but in fact would be different.) Then an approximation to a smooth trinoid made with N. Schmitt's CMCLab software and a true discrete trinoid found in T. Hoffmann's thesis [8] are shown.

This can already be seen in the discrete minimal catenoids coming from each way, as these two types of catenoids really do not coincide.

The integrable systems viewpoint for smooth CMC surfaces itself leads to both approaches for discretizing. There is a method, called the DPW method after its founders Dorfmeister, Pedit and Wu [5], that is based on integrable systems methods and produces smooth CMC surfaces (or more generally harmonic maps into symmetric spaces). Central to the method is an introduction of a spectral parameter λ lying in the unit circle S^1 in the complex plane. In this method one uses techniques in integrable systems theory, dating back at least to Kričever [10], to construct an object called an *extended frame* depending on λ, from which a CMC surface can be constructed. In fact, any CMC surface can be constructed this way. One is implicitly finding a solution to the sinh-Gordon equation, but the beauty of the method is that the sinh-Gordon equation itself is essentially bypassed and the surface is constructed without needing to know anything specific about that difficult-to-find sinh-Gordon equation solution.

Now one can take either of the two approaches described in this article for discretizing the DPW method. The first approach is to find discrete approximations to the desired smooth surface. In this case the smooth surface is doubly an infinite dimensional problem, because you have both the surface parameter and the spectral parameter. The surface parameter can be discretized in the usual way, and the spectral parameter λ can be discretized by chopping away all but a finite number of terms in the Fourier series for the extended frame. What is left is a finite dimensional problem, and this is exactly what is solved numerically in the CMCLab program by N. Schmitt. The second approach is to create a discrete formulation of the DPW method that preserves integrable systems properties. This approach has been taken by T. Hoffman [8].

By the above avenues, and other avenues as well, computers and discrete methods and integrable systems methods have come to play a central role in surface theory.

The DPW method: For those who are interested, we close with a more technical description of the DPW recipe for constructing any nonminimal CMC surface in R^3. On a Riemann surface Σ with local complex coordinates z, we define a *holomorphic potential* as a trace-free matrix-valued λ-dependent 1-form, $\lambda \in S^1$,

$$\xi = \begin{pmatrix} \sum_{j=0}^{\infty} c_j(z)\lambda^j & \sum_{j=-1}^{\infty} a_j(z)\lambda^j \\ \sum_{j=0}^{\infty} b_j(z)\lambda^j & -\sum_{j=0}^{\infty} c_j(z)\lambda^j \end{pmatrix} dz \ ,$$

where the $a_j dz, b_j dz, c_j dz$ are all holomorphic 1-forms defined on Σ, and a_{-1} is never zero. Choose an SL_2C-valued solution ϕ of

$$d\phi = \phi\xi \ ,$$

analytic in λ, and write $\phi = FB$ (this is Iwasawa splitting) so that F is SU_2-valued for all $\lambda \in S^1$ and B extends holomorphically to $\{\lambda \in C \mid |\lambda| \leq 1\}$ and $B|_{\lambda=0}$ is upper-triangular. Although ϕ is holomorphic in z, F and B are only real-analytic in z. We call F an *extended frame* because it is represents a framing for a CMC surface at any $\lambda \in S^1$. Then we insert F into the Sym-Bobenko formula, choosing $\lambda = 1$,

$$f = -2iH^{-1} \left[\lambda\partial_\lambda F \cdot F^{-1}\right]_{\lambda=1} \ ,$$

which is of the form

$$f = \frac{-i}{2} \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \ ,$$

for real-valued functions $x_1 = x_1(z, \bar{z}), x_2 = x_2(z, \bar{z}), x_3 = x_3(z, \bar{z})$, and then the theory behind the DPW method implies that

$$\Sigma \ni z \mapsto (x_1, x_2, x_3) \in R^3$$

becomes a conformal parametrization of a CMC H surface.

Among the simple examples of holomorphic potentials, for the sphere, cylinder and Delaunay surfaces, respectively, are

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz \ , \quad \Sigma = C \ ,$$

$$\xi = \frac{1}{4} \begin{pmatrix} 0 & \lambda^{-1} + 1 \\ 1 + \lambda & 0 \end{pmatrix} dz \ , \quad \Sigma = C \setminus \{0\} \ ,$$

$$\xi = \xi = \begin{pmatrix} 0 & s\lambda^{-1} + t \\ s\lambda + t & 0 \end{pmatrix} \frac{dz}{z} \ , \quad \Sigma = C \setminus \{0\} \ ,$$

for $s, t \in R \setminus \{0, 1/4\}$ and $s + t = 1/2$.

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