Conserved quantities in the theory of discrete surfaces

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1 Background

Suppose you are given a simple first order smooth ordinary differential equation with a given initial condition. If you cannot write down its solution explicitly, you might find a discrete approximate solution by using the Euler or Runga-Kutta algorithm, just to have some initial idea how the smooth solution behaves. In this case, your interest in the approximate solution is only as a stepping stone for understanding the smooth true solution. We can think of the equation (i.e. the algorithm) for the discrete approximate solution as a finite dimensional problem because the full space of objects (a vector space of discrete functions) that can be inserted to test for validity in the equation is finite dimensional. Likewise, we can call the smooth differential equation an infinite dimensional problem (this might be somewhat unconventional), because the objects insertable into the equation form an infinite dimensional vector space.

Or you might instead look at a related ordinary difference equation, with little concern that the resulting discrete solution approximates the smooth solution, and rather be more concerned that the difference equation maintains some property found in the smooth differential equation that you deem important. In this case, as your primary interest is the "finite dimensional" difference equation situation itself, you might discard the smooth equation altogether, or you might acknowledge the existence of the smooth equation but regard it only as an incidental limiting case of the difference equation you care much more about.

Both approaches are of interest, and are now common in surface theory, though usually involving partial differential equations, not ordinary ones. Discrete analogs of smooth minimal and constant mean curvature surfaces are being studied. But there is no single definitive way to define these analogs; the definition one chooses depends on which properties of smooth minimal and constant mean curvature surfaces one wishes to emulate in the discrete case.

A good example of the first approach is the Costa surface, with the following story: The simplest example of a minimal surface in R^3 is the plane, and two other rather simple examples are: 1) the catenoid, a surface of revolution produced by revolving a catenary and parametrized by

$$\{(\cosh u \cos v, \cosh u \sin v, u) \in R^3 \,|\, u \in R, v \in [0, 2\pi)\} \ ,$$

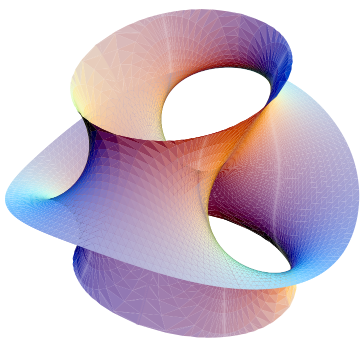


Figure 1: The Costa surface.

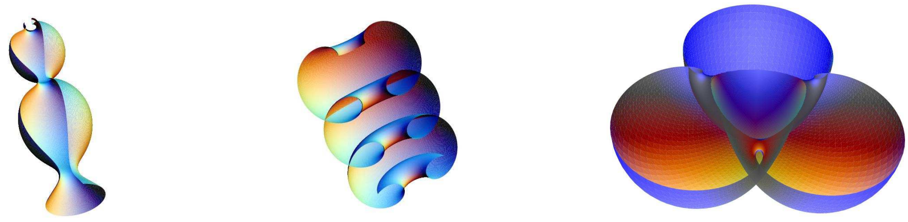


Figure 2: Cut-aways of three constant mean curvature surfaces, a Delaunay unduloid, a Delaunay nodoid and a Wente torus, in R^3. The first two are surfaces of revolution.

where R denotes the real numbers, and 2) the helicoid, foliated by straight lines and parametrized by

$$\{(\sinh u \cos v, \sinh u \sin v, v) \in R^3 \mid u, v \in R\} \ .$$

The famous Weierstrass representation says that all minimal surfaces can be locally parametrized by pairs of meromorphic functions f, g defined on Riemann surfaces Σ with local complex coordinates z, by using path integrals:

$$\mathrm{Re} \int_{z_0}^{z} (1-g^2, i+ig^2, 2g) f dz \ , \quad i = \sqrt{-1} \ .$$

The Costa surface, found in 1984 by Costa [5] is a complete minimal surface homeomorphic to a torus minus three points. It has the Weierstrass data

$$\Sigma = \{(z,w) \in (C \cup \{\infty\})^2 \mid w^2 = z(z^2-1)\} \setminus \{(-1,0),(1,0),(\infty,\infty)\} \ ,$$

and

$$g = B/w \ , \quad f = w/(z^2-1) \ ,$$

where B is the constant

$$B = \sqrt{2 \int_0^1 \left(\frac{t}{1-t^2} \right)^{1/2} dt \Bigg/ \int_0^1 \frac{dt}{t(1-t^2)^{1/2}}} \ .$$

There had been a long standing conjecture that the only complete embedded minimal surfaces with finite topology in R^3 are the plane and catenoid and helicoid. In 1985, Hoffman and Meeks [9] confirmed that the Costa surface is a counterexample by proving it is embedded, but only after numerics led them to see that the surface possessed certain lines and planes of symmetry that were useful for the proof. Though their final proof used no numerics, the numerics helped them to find it.

There are also examples of this first approach amongst the constant mean curvature (CMC) surfaces in R^3. Simple examples of these surfaces are the round sphere and round cylinder. Less trivial examples are Delaunay surfaces of revolution, parametrizable explicitly in terms of the nonconstant periodic Jacobi elliptic function $v(x)$ satisfying

$$(v')^2 = -(v^2-4s^2)(v^2-4t^2) \ , \quad v(0) = 2|t| \ ,$$

with $s, t \in R \setminus \{0\}$, $s \neq t$ and $s+t = 1/2$, and the elliptic integral of the third kind

$$\int_0^x \frac{4st}{4st + v^2(\rho)} d\rho \ .$$

Hopf [11] asked whether any compact CMC surface without boundary in R^3 must be a round sphere. He proved it is true when the surface is simply connected. Alexandrov proved it when the surface is embedded, using the maximum principle for second-order elliptic differential equations. However, Wente [17] showed it is false in general, by finding compact nonembedded

CMC surfaces in R^3 without boundary and of genus 1. These tori can be described in terms of Jacobi elliptic functions and integrals, as the Delaunay surfaces are.

As an example of the first approach, Abresch saw from numerical experimentation that one family of curvature lines on the Wente tori appeared to be planar curves. He then proceeded to mathematically prove the existence of such CMC tori [1]. After that, Spruck [16] showed that the tori found by Abresch are exactly the same collection of surfaces as found by Wente.

To apply the second approach, on the other hand, one must decide what properties of smooth CMC surfaces one would like to see preserved in the discrete CMC surfaces, and different choices of those properties result in genuinely different theories.

One choice, following the definitions given above, is to demand that the discrete CMC surfaces also locally minimize area with respect to variations of the surface that preserve volume to each side. For this, one would choose the discrete surfaces to be triangulated, i.e. as surfaces made by gluing triangles together along edges, and then consider variations that continuously move the vertices while preserving the simplicial structure. If any variation that moves just one interior vertex and preserves the volume to each side of the surface will never decrease area, we can say that this discrete surface is of constant mean curvature. See works of K. Polthier [14], [15], and the Surface Evolver by K. Brakke.

But another choice of the property to be preserved, now often used, is as follows: The governing equation, i.e. the Gauss equation, when using conformal coordinates, for the local existence of a CMC surface is the sinh-Gordon equation

$$\partial_{\bar{z}}\partial_z u + \sinh u = 0 \ ,$$

where z is a local complex coordinate on a Riemann surface. (Let us ignore umbilic points here. Anyway, these umbilics are isolated on any CMC surface other than the round sphere.)

So we can now obtain discrete CMC surfaces via a discretization of that integrable system (the sinh-Gordon equation). In this case, the area-minimizing property has been lost, so there is no reason to think about variations of the surface, and so one can consider the surfaces as a mesh of planar quadrilaterals (for which one cannot freely move the vertices, as planarity will then be lost) rather than of triangles. Perhaps the first breakthrough in this field was by Wunderlich [18], with further developments by the TU-Berlin geometry group.

At first, preserving the area-minimizing property for the discrete surfaces might seem preferable to preserving a relationship with integrable systems, as the former property is fundamental to how the smooth surfaces are defined, while the latter property is something that one later discovers in their mathematical structure. The first way is clearly important, but there are good reasons for considering the second way too. The second way, by preserving relations to integrable systems, preserves much of the interesting underlying mathematical structure [2], [8]. (For example, only the second way gives discrete versions of the Bianchi permutability theorem, and discrete transformations of Backlund, Darboux or Ribaucour type.)

Interestingly, it seems that one cannot simultaneously preserve the area-minimizing property and the relationships to integrable systems. This can already be seen in the discrete minimal catenoids coming from each way, as these two types of catenoids really do not coincide.

The integrable systems viewpoint for smooth CMC surfaces itself leads to both approaches for discretizing. There is a method, called the DPW method after its founders Dorfmeister, Pedit and Wu [7], that is based on integrable systems methods and produces smooth CMC surfaces (or more generally harmonic maps into symmetric spaces). Central to the method is an introduction of a spectral parameter λ lying in the unit circle S^1 in the complex plane. In this method one uses techniques in integrable systems theory, dating back at least to Kričever [12], to construct an object called an *extended frame* depending on λ, from which a CMC surface can be constructed. In fact, any CMC surface can be constructed this way. One is implicitly finding a solution to the sinh-Gordon equation, but the beauty of the method is that the sinh-Gordon equation itself is essentially bypassed and the surface is constructed without needing to know anything specific about that solution.

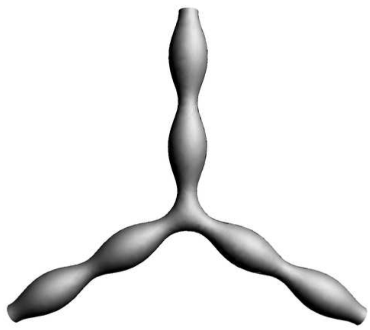


Figure 3: Graphics of a smooth trinoid made with N. Schmitt's CMCLab software.

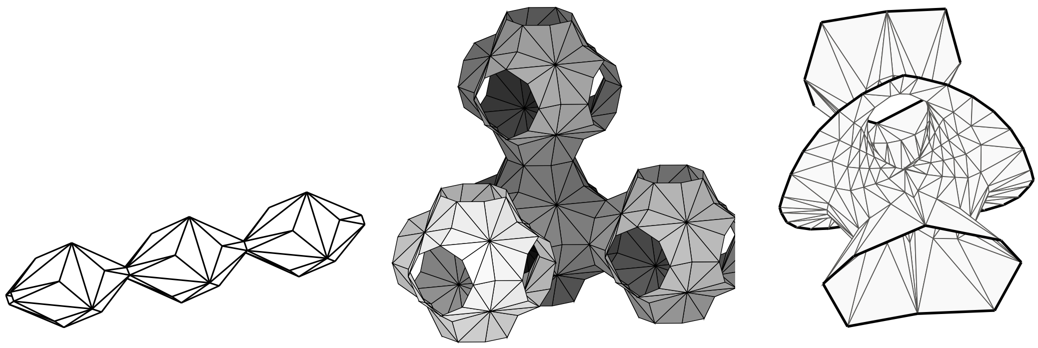


Figure 4: Discrete surfaces made from a variational viewpoint. Left: a discrete CMC Delaunay surface. Middle: a discrete triply-periodic Schwarz P minimal surface. Right: a discrete minimal surface similar to the smooth Costa minimal surface.

Now one can take either of the two approaches described here for discretizing the DPW method. The first approach is to find discrete approximations to the desired smooth surface. In this case the smooth surface is doubly an infinite dimensional problem, because you have both the surface parameter and the spectral parameter. The surface parameter can be discretized in the usual way, and the spectral parameter λ can be discretized by chopping away all but a finite number of terms in the Fourier series for the extended frame. What is left is a finite dimensional problem, and this is exactly what is solved numerically in the CMCLab program by N. Schmitt. The second approach is to create a discrete formulation of the DPW method that preserves integrable systems properties. This approach has been taken by T. Hoffman [10].

By the above avenues, and others as well, computers and discrete methods and integrable systems methods have come to play a central role in surface theory.

We now describe the DPW recipe for constructing any nonminimal CMC surface in R^3. On a Riemann surface Σ with local complex coordinates z, we define a *holomorphic potential* as a trace-free matrix-valued λ-dependent 1-form, $\lambda \in S^1$,

$$\xi = \begin{pmatrix} \sum_{j=0}^{\infty} c_j(z)\lambda^j & \sum_{j=-1}^{\infty} a_j(z)\lambda^j \\ \sum_{j=0}^{\infty} b_j(z)\lambda^j & -\sum_{j=0}^{\infty} c_j(z)\lambda^j \end{pmatrix} dz \ ,$$

where the $a_j dz, b_j dz, c_j dz$ are all holomorphic 1-forms defined on Σ, and a_{-1} is never zero. Choose an SL_2C-valued solution ϕ of

$$d\phi = \phi\xi \ ,$$

analytic in λ, and write $\phi = FB$ (this is Iwasawa splitting) so that F is SU_2-valued for all $\lambda \in S^1$ and B extends holomorphically to $\{\lambda \in C \,|\, |\lambda| \leq 1\}$ and $B|_{\lambda=0}$ is upper-triangular. Although ϕ is holomorphic in z, F and B are only real-analytic in z. We call F an *extended frame* because it is represents a framing for a CMC surface at any $\lambda \in S^1$. Then we insert F

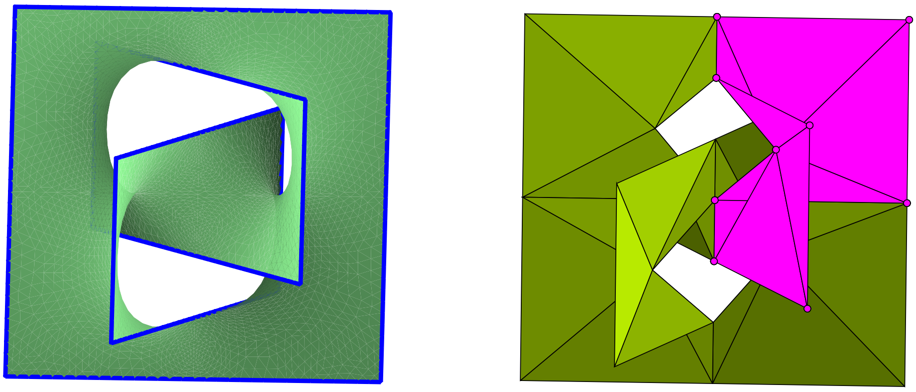


Figure 5: Smooth and discrete versions of the triply-periodic Fischer-Koch type minimal surface. The discrete version is made from a variational viewpoint.

into the Sym-Bobenko formula, choosing $\lambda = 1$,

$$f = -2iH^{-1}\left[\lambda \partial_\lambda F \cdot F^{-1}\right]_{\lambda=1} ,$$

which is of the form

$$f = \frac{-i}{2}\begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} ,$$

for real-valued functions $x_1 = x_1(z,\bar{z}), x_2 = x_2(z,\bar{z}), x_3 = x_3(z,\bar{z})$, and then one can prove that

$$\Sigma \ni z \mapsto (x_1, x_2, x_3) \in R^3$$

becomes a conformal parametrization of a CMC H surface.

Among the simple examples of holomorphic potentials, for the sphere, cylinder and Delaunay surfaces, respectively, are

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz , \quad \Sigma = C ,$$

$$\xi = \frac{1}{4}\begin{pmatrix} 0 & \lambda^{-1} + 1 \\ 1 + \lambda & 0 \end{pmatrix} dz , \quad \Sigma = C \setminus \{0\} ,$$

$$\xi = \xi = \begin{pmatrix} 0 & s\lambda^{-1} + t \\ s\lambda + t & 0 \end{pmatrix} \frac{dz}{z} , \quad \Sigma = C \setminus \{0\} ,$$

for $s, t \in R \setminus \{0, 1/4\}$ and $s + t = 1/2$.

2 A conserved quantities approach to smooth CMC surfaces

In the next section we introduce an approach to discrete CMC surfaces coming from joint work with F. Burstall, U. Hertrich-Jeromin, and S. Santos. But to motivate that discussion, in this section we first explain a result of Burstall and Calderbank [4] for the case of smooth CMC surfaces. We begin by describing the 3-dimensional space forms using the 5-dimensional Minkowski space $R^{4,1}$.

Minkowski 5-space. We give a 2×2 matrix formulation for Minkowski 5-space. Let H denote the quaternions and $\operatorname{Im} H$ the imaginary quaternions.

$$R^{4,1} = \left\{ X = \begin{pmatrix} x & x_\infty \\ x_0 & -x \end{pmatrix} \middle| \; x \in \operatorname{Im} H, x_0, x_\infty \in R \right\}$$

with Minkowski metric $\langle X, Y\rangle$ such that $\langle X, Y\rangle \cdot I = -\frac{1}{2}(XY + YX)$, I = identity matrix. This metric has signature $(+,+,+,+,-)$ with respect to the (orthonormal) basis

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we set $x_4 = \frac{1}{2}(x_\infty - x_0)$, $x_5 = \frac{1}{2}(x_\infty + x_0)$, we can write X as

$$X = x_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + x_2 \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} + x_3 \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} + x_4 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $x = x_1 i + x_2 j + x_3 k$, and then we have the correspondence $X \leftrightarrow (x_1, x_2, x_3, x_4, x_5)$ to the more usual way

$$\{(x_1, x_2, x_3, x_4, x_5) \in R^5 \,|\, ||(x_1, x_2, x_3, x_4, x_5)|| = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2\}$$

of denoting $R^{4,1}$. The 4-dimensional light cone is

$$L^4 = \{X \in R^{4,1} \,|\, ||X||^2 = 0\}.$$

We can make the 3-dimensional space forms as follows: A space form M is $M = L^4 \cap \{X \,|\, \langle X, Q\rangle = -I\}$ for any nonzero $Q \in R^{4,1}$. It will turn out that M has curvature κ, where $Q^2 = \kappa \cdot I$, so without loss of generality we can obtain any space form by choosing

$$Q = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix}, \tag{1}$$

and then

$$M = \left\{ X = \frac{2}{1 - \kappa x^2} \cdot \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} \right\},$$

which is equivalent to $\{(x_1, x_2, x_3) \in R^3 \cup \{\infty\} \,|\, x_1^2 + x_2^2 + x_3^2 \neq -\kappa^{-1}\}$, where $x = x_1 i + x_2 j + x_3 k \in \operatorname{Im} H$. Note that when $\kappa < 0$, M becomes two copies of hyperbolic 3-space with sectional curvature κ.

The tangent space of M at X is

$$T_X M = \left\{ \mathcal{T}_a = \frac{2}{(1 - \kappa x^2)^2} \cdot \begin{pmatrix} a + \kappa x a x & -xa - ax \\ \kappa(xa + ax) & -a - \kappa x a x \end{pmatrix} \right\},$$

for $a \in \operatorname{Im} H$. When $X = X(t) \in M$ is a smooth function of a real variable t, and when $'$ denotes differentiation with respect to t, we have

$$X' = \mathcal{T}_{x'}.$$

A computation gives

$$\langle \mathcal{T}_a, \mathcal{T}_b\rangle = \frac{-4}{(1 - \kappa x^2)^2} \operatorname{Re}(ab), \tag{2}$$

$$||\mathcal{T}_a|| = 1 \Leftrightarrow |a| = \tfrac{1}{2}|1 - \kappa x^2|.$$

Also,

$$X'' = \mathcal{T}_{\frac{2\kappa(xx' + x'x)}{1 - \kappa x^2} \cdot x' + x''} + \frac{4(x')^2}{(1 - \kappa x^2)^2} \cdot \begin{pmatrix} \kappa x & -1 \\ \kappa & -\kappa x \end{pmatrix}. \tag{3}$$

Note that generally X'' is not contained in $T_X M$.

The following lemma follows from (2).

Lemma 1. *The M determined by the Q in* (1) *has constant sectional curvature κ.*

We see from (2) that the collection of M given by the above choice (1) for Q, for various κ, are all conformally equivalent (or Moebius equivalent).

Smooth surfaces in space forms. Let

$$x = x_1(u,v)i + x_2(u,v)j + x_3(u,v)k \approx X \in M$$

be a surface in M. Assume (u,v) is a conformal curvature-line coordinate system (every CMC surface can be parametrized this way). We call such coordinates *isothermic* coordinates.

Note that x_1, x_2 and x_3 can be chosen before the space form M is chosen, and only once M (and hence κ) is chosen do we know the form of X. In particular, the surface can be defined before the space form is chosen. Because we will always choose Q as in (1), we will indicate this by denoting M as M_κ, with the subscript κ.

We let n denote the unit normal vector for x, once M_κ is chosen. n_0 denotes the unit normal with respect to Euclidean 3-space M_0, where $\kappa = 0$. We sometimes write H_κ for the mean curvature of the surface x with respect to the space form M_κ, to denote that the mean curvature depends on the choice of space form. H_0 is the mean curvature in the case of Euclidean 3-space M_0.

Lemma 2. *The mean curvature $H = H_\kappa$ of x with respect to the space form M_κ given by Q as in* (1)*, with $\triangle x = \partial_u\partial_u x + \partial_v\partial_v x$, is*

$$H = \tfrac{-1}{2}|x_u|^{-2}\,\mathrm{Re}\{\triangle x \cdot n\} - \frac{\kappa}{1-\kappa x^2}(xn + nx) =$$

$$= \tfrac{-1}{2}(1-\kappa x^2)|x_u|^{-2}\,\mathrm{Re}\{\triangle x \cdot n_0\} - \kappa(xn_0 + n_0x) =$$

$$(1-\kappa x^2)H_0 - \kappa(xn_0 + n_0 x)\ .$$

Then H is constant exactly when $\partial_u H = \partial_v H = 0$, which is equivalent to

$$(\partial_u H_0)\cdot(1-\kappa x^2) = \kappa\tfrac{k_2-k_1}{2}\partial_u(x^2)\ ,\quad (\partial_v H_0)\cdot(1-\kappa x^2) = \kappa\tfrac{k_1-k_2}{2}\partial_v(x^2)\ , \tag{4}$$

where the $k_j \in R$ are the principal curvatures with respect to the Euclidean space form M_0, i.e. $\partial_u n_0 = -k_1\partial_u x$ and $\partial_v n_0 = -k_2\partial_v x$.

Proof. Letting x_{1u} denote $\frac{d}{du}(x_1)$, and similarly taking other notations, the unit normal vector to the surface is $\mathcal{T}_n$, where $n = (1-\kappa x^2)n_0$ and

$$n_0 = \frac{1}{2}\cdot\frac{(x_{2u}x_{3v} - x_{3u}x_{2v})i + (x_{3u}x_{1v} - x_{1u}x_{3v})j + (x_{1u}x_{2v} - x_{2u}x_{1v})k}{\sqrt{(x_{2u}x_{3v} - x_{3u}x_{2v})^2 + (x_{3u}x_{1v} - x_{1u}x_{3v})^2 + (x_{1u}x_{2v} - x_{2u}x_{1v})^2}}\ .$$

The first fundamental form (g_{ij}) satisfies $\langle \mathcal{T}_{x_u}, \mathcal{T}_{x_v}\rangle = 0 = g_{12} = g_{21}$, and

$$g_{11} = \langle \mathcal{T}_{x_u}, \mathcal{T}_{x_u}\rangle = \frac{4|x_u|^2}{(1-\kappa x^2)^2} = \frac{4|x_v|^2}{(1-\kappa x^2)^2} = \langle \mathcal{T}_{x_v}, \mathcal{T}_{x_v}\rangle = g_{22}\ .$$

Then using (3), with the symbol $'$ denoting either ∂_u or ∂_v, we have (where the superscript "T" denotes the part of a vector tangent to $T_X M$)

$$b_{11} = \langle X_{uu}^T, \mathcal{T}_n\rangle = \langle X_{uu}, \mathcal{T}_n\rangle = \frac{-4}{(1-\kappa x^2)^2}\,\mathrm{Re}\{x_{uu}\cdot n\} + \frac{4\kappa x_u^2}{(1-\kappa x^2)^3}(xn+nx)\ ,$$

$$b_{12} = b_{21} = \langle X_{uv}^T, \mathcal{T}_n\rangle = \langle X_{uv}, \mathcal{T}_n\rangle = 0\ ,$$

$$b_{22} = \langle X_{vv}^T, \mathcal{T}_n\rangle = \langle X_{vv}, \mathcal{T}_n\rangle = \frac{-4}{(1-\kappa x^2)^2}\,\mathrm{Re}\{x_{vv}\cdot n\} + \frac{4\kappa x_v^2}{(1-\kappa x^2)^3}(xn+nx)\ .$$

The result follows, using $H_0 = (k_1+k_2)/2$. □

We now define the Christoffel transformation x^*, which for a CMC surface in R^3 gives the parallel CMC surface. Let x give a surface in R^3 with mean curvature H_0 and unit normal n_0. The Christoffel transformation x^* satisfies that

- x^* is defined on the same domain as x,
- x^* has the same conformal structure as x,
- x and x^* have opposite orientations,
- and x and x^* have parallel tangent planes at corresponding points.

This definition above turns out to be equivalent to the following definition, and the existence of the integrating factor ρ below is equivalent to the existence of isothermic coordinates. Then, once we have x^*, we will see that we can take x^* so that $dx^* = x_u^{-1}du - x_v^{-1}dv$.

Definition 1. *A Christoffel transformation x^* of x in R^3 is such that $dx^* = \rho(dn_0 + H_0 dx)$ for some real-valued function ρ on the surface x (here x^* is determined only up to translations and homotheties).*

Lemma 3. *x^* exists if and only if x is isothermic.*

Proof. We prove only one direction here. Assume x is isothermic, and take isothermic coordinates u, v for x, so $x_{uv} = Ax_u + Bx_v$ for some A, B. Then

$$d(x_u^{-1}du - x_v^{-1}dv) = 16g_{11}^{-2}(x_u x_{uv} x_u + x_v x_{uv} x_v) du \wedge dv = 0 .$$

This implies that there exists an x^* such that

$$dx^* = x_u^{-1}du - x_v^{-1}dv .$$

Also,

$$dn_0 + H_0 dx = \tfrac{1}{8}(b_{11} - b_{22})(x_u^{-1}du - x_v^{-1}dv) ,$$

implying that x^* is a Christoffel transform. □

Corollary 1. *Christoffel transformations can be taken as solutions of $dx^* = x_u^{-1}du - x_v^{-1}dv$.*

As a result of the above corollary, we can now simply take the definition of x^* as follows:

Definition 2. *The Christoffel transformation of x is any x^* (defined in R^3 up to translation) such that $dx^* = x_u^{-1}du - x_v^{-1}dv$.*

Lemma 4.

$$dx^* = \frac{2}{(k_1 - k_2)|x_u|^2}(dn_0 + H_0 dx) .$$

Proof.

$$\left(\frac{2}{(k_1 - k_2)|x_u|^2}(dn_0 + H_0 dx) - x_u^{-1}du + x_v^{-1}dv \right) |x_u|^2 =$$

$$= \frac{2}{k_1 - k_2}(-k_1 x_u du - k_2 x_v dv + \tfrac{k_1+k_2}{2}(x_u du + x_v dv)) + x_u du - x_v dv = 0 .$$

□

The following corollary shows that the Christoffel transform is what it should be when the ambient space is R^3, and this surface is CMC, i.e. it gives the parallel CMC surface.

Corollary 2. *If H_0 is constant for the surface x in R^3, then x^* is equal to a real constant times $x + H_0^{-1} n_0$.*

Proof. Because H_0 is constant and we have isothermic coordinates, the $\hat{Q}$ in the Hopf differential $\hat{Q}(d(u + iv))^2$ is real. Because $\hat{Q}$ is holomorphic with respect to $u + iv$, it is therefore a real constant. So

$$\hat{Q} = \langle n_0, x_{uu} - x_{vv} \rangle = (k_1 - k_2)|x_u|^2$$

is constant. We then apply Lemma 4. □

In the next definition, the nonzero real constant c can be chosen freely, and we are once again considering general space forms M.

Definition 3. *For some $c \in R \setminus \{0\}$, we set $\tau = c\begin{pmatrix} xdx^* & -xdx^*x \\ dx^* & -dx^*x \end{pmatrix}$. If there exist smooth Q and Z in $R^{4,1}$ depending on (u,v) such that*

$$d(Q+\lambda Z) = (Q+\lambda Z)\lambda\tau - \lambda\tau(Q+\lambda Z) \tag{5}$$

holds for all $\lambda \in R$, then we call $Q+\lambda Z$ a linear conserved quantity *of x.*

Some properties of linear conserved quantities are immediate. For example, Q and Z^2 are constant, $X\tau = \tau X = 0$, $X \perp Z$ and $X \perp dZ$. We now show these properties:

Lemma 5. *Q is constant.*

Proof. Set $\lambda = 0$ in the conserved quantity equation (5). □

Lemma 6. *$X\tau = \tau X = 0$.*

Proof.

$$X\tau = \frac{2c}{1-\kappa x^2}\begin{pmatrix} x \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -x \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} dx^* \begin{pmatrix} 1 & -x \end{pmatrix} = 0 \ ,$$

since

$$\begin{pmatrix} 1 & -x \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = 0 \ .$$

Similarly one can show $\tau F = 0$. □

Lemma 7. *If $Q+\lambda Z$ is a linear conserved quantity, then Z^2 is constant.*

Proof. First note that $d(Z^2) = Z\cdot dZ + dZ\cdot Z = Z(Q\tau - \tau Q) + (Q\tau - \tau Q)Z = (QZ+ZQ)\tau - \tau(QZ+ZQ)$, since $Z\tau = \tau Z$. Because $QZ+ZQ$ is real, we have $d(Z^2) = 0$. □

Lemma 8. *X is perpendicular to both Z and dZ.*

Proof. $XZ+ZX$ is a real multiple of the identity, and is zero because $\tau(XZ+ZX) = \tau ZX = Z\tau X = 0$. Thus, $X \perp Z$. Next, $X\cdot dZ + dZ\cdot X = X(Q\tau - \tau Q) + (Q\tau - \tau Q)X = XQ\tau - \tau QX = (-QX - I)\tau - \tau(-XQ - I) = -\tau + \tau = 0$. Thus $X \perp dZ$. □

Properties like these will be utilized to prove the following theorem:

Theorem 1. *[4] The surface x is constant mean curvature in a space form M (produced by $Q \neq 0$) if and only if there exists (for that Q) a linear conserved quantity $Q+\lambda Z$.*

Furthermore, when x is not totally umbilic, then Z is unique with positive norm. In fact, in the following proof we can see that Z is uniquely determined by the mean curvature and normal vector, so Z represents the central sphere congruence.

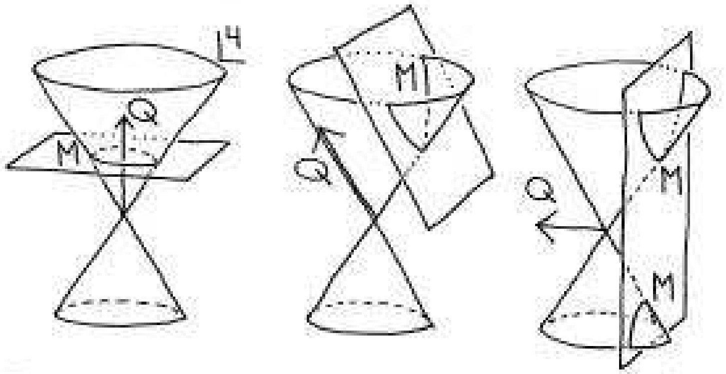
Proof. In the case that x is part of a sphere, then $dx^* = 0$, so $\tau = 0$, and so we can take $Z = 0$. So we now assume x is not totally umbilic.

Assume that x has a linear conserved quantity. We can take Q as in (1), and denote the components of Z by

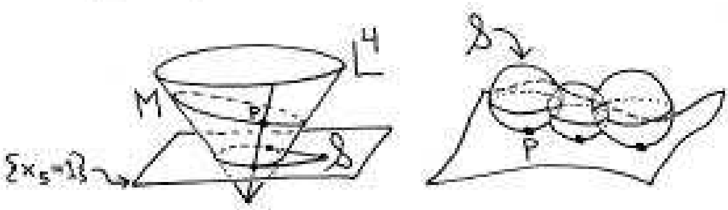
$$Z = \begin{pmatrix} z & z_\infty \\ z_0 & -z \end{pmatrix} \ .$$

The above lemmas tell us that $XZ+ZX = 0$ and $X \perp dZ$, which, respectively, imply

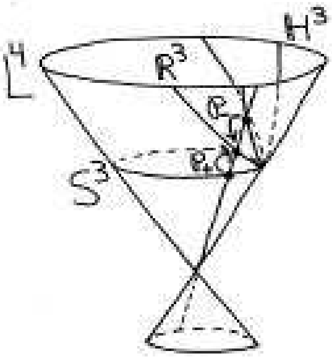
$$xz - x^2 z_0 + zx + z_\infty = 0 \quad \text{and} \quad x\,dz - x^2 dz_0 + dz\,x + dz_\infty = 0 \ .$$



Three choices of K (K > 0, K = 0, K < 0) giving the space forms S^3, $\mathbb{R}^3$ and (two copies of) $\mathbb{H}^3$.



A typical picture of an envelope (with spheres labeled δ) on the right, and the corresponding picture in the $\mathbb{R}^{4,1}$ model on the left.



P_+ and P_- are conformal maps from S^3 and $\mathbb{H}^3$ to $\mathbb{R}^3$, showing that S^3, $\mathbb{R}^3$ and $\mathbb{H}^3$ are Moebius equivalent.

Differentiating the first of these two equations, and then applying the second one, we have

$$dx \cdot z - (xdx + dx \cdot x)z_0 + zdx = 0 \ ,$$

which implies

$$z = z_0 \cdot x + h \cdot n_0$$

for some real-valued function h. Then

$$x(z_0x + hn_0) - x^2 z_0 + (z_0x + hn_0)x + z_\infty = hxn_0 + z_0x^2 + hn_0x + z_\infty = 0 \ ,$$

so

$$z_\infty = -h(xn_0 + n_0x) - z_0x^2 \ .$$

Thus

$$Z = z_0 \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} + h \begin{pmatrix} n_0 & -n_0x - xn_0 \\ 0 & -n_0 \end{pmatrix} \ .$$

Because Z^2 is constant,

$$(z_0x + hn_0)^2 - z_0h(xn_0 + n_0x) - z_0^2x^2 = -h^2$$

is constant, and so h is constant, and then also $|Z|$ is constant and nonnegative. A direct computation, using $n_0\, dx^* + dx^*\, n_0 = 0$, now shows that $Z\tau = \tau Z$, so the condition $Z\tau = \tau Z$ coming from Equation (5) provides no extra information. The relation $dZ = Q\tau - \tau Q$ from (5) gives that

$$dz_0 = c\kappa(x \cdot dx^* + dx^* \cdot x) \text{ and } dz_0 \cdot x + z_0dx + hdn_0 = c(dx^* + \kappa x dx^* x) \ .$$

These two equations tell us that

$$h = \frac{2c(1-\kappa x^2)}{x_u^2(k_2 - k_1)} \ , \tag{6}$$

which we know to be constant, and that

$$z_0 = \tfrac{1}{2}h(k_2 + k_1) = h \cdot H_0 \ . \tag{7}$$

Equations (6) and (7) tell us that (4) holds, and so H_κ is constant. One direction of the theorem now follows.

To prove the other direction, assume that x is a CMC surface with isothermic coordinate $z = u + iv$, then the Hopf differential is a constant multiple of dz^2, so

$$b_{11} - b_{22} = \frac{4x_u^2(k_2 - k_1)}{1 - \kappa x^2}$$

is constant, and so

$$h = \frac{2c(1-\kappa x^2)}{x_u^2(k_2 - k_1)} \ , \quad c \in R \setminus \{0\} \ ,$$

is also constant. Take Q as in (1), and then take

$$Z = hH_0 \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} + h \begin{pmatrix} n_0 & -xn_0 - n_0x \\ 0 & -n_0 \end{pmatrix} \ .$$

Then set the candidate for the conserved quantity to be $P = Q + \lambda Z$, where $dx^* = x_u^{-1}du - x_v^{-1}dv$, and τ is as in Definition 3. Then a computation gives $dP + \lambda\tau P - P\lambda\tau = 0$, by Equation (4). □

We now explain the conserved quantity equation in terms of the Calapso transformation, in order to motivate a definition used in the discrete setting.

Definition 4. *Let x be a surface with isothermic coordinates. A Calapso transformation $T \in Mob(3)$ is a solution of*

$$T^{-1}dT = \lambda\tau \ .$$

Then the transformation $x \to Tx$ is a Calapso transformation. (We can also call it a T-transform or "conformal deformation".)

The Calapso transformation is classical, and was studied by Calapso, Bianchi and Cartan. It preserves the conformal structure and is thus of interest in Moebius geometry. However, in the case that the starting surface is CMC, it is the same as the Lawson correspondence, which is quite important in the differential geometry of CMC surfaces.

Lemma 9. *If x is isothermic, then the Calapso transform exists.*

Proof. The system

$$T^{-1}T_u = \lambda U \ , \quad U = \begin{pmatrix} x \\ 1 \end{pmatrix} x_u^{-1} \begin{pmatrix} 1 & -x \end{pmatrix} \ ,$$

$$T^{-1}T_v = \lambda V \ , \quad V = -\begin{pmatrix} x \\ 1 \end{pmatrix} x_v^{-1} \begin{pmatrix} 1 & -x \end{pmatrix}$$

has a solution if and only if $UV - VU + V_u - U_v = 0$, and this equation holds precisely because of the conditions for isothermicness, that is

$$x_u^2 = x_v^2 \ , \quad x_u x_v + x_v x_u = 0 \ , \quad x_{uv} = Ax_u + Bx_v$$

for some functions A, B. □

Now if we set $P = Q + \lambda Z$, then

$$dP + \lambda\tau P - P\lambda\tau = 0$$

if and only if $dP + T^{-1}dT \cdot P - P \cdot T^{-1}dT = 0$ if and only if

$$d(TPT^{-1}) = 0$$

if and only if TPT^{-1} is constant. It is this last condition of TPT^{-1} being constant that we will use to define discrete CMC surfaces, just as it defines smooth CMC surfaces, by Theorem 1.

Darboux transformations. For smooth surfaces, a Darboux transform is one such that

- there exists an envelope of spheres between the original surface and the transform,
- the envelope (i.e. the transform) preserves curvature lines, and
- the transform preserves conformality.

3 A conserved quantities approach to discrete CMC surfaces

Our purpose in this section is to present a definition for discrete constant mean curvature (CMC) H surfaces in any of the three space forms Euclidean 3-space R^3, spherical 3-space S^3 and hyperbolic 3-space H^3. This new definition is equivalent to the previously known definitions [2] in the case of R^3. It also satisfies a Calapso transformation relation (the Lawson correspondence), suggesting the definition is also natural for the space form S^3, and for CMC surfaces with $H \geq 1$ in H^3. The definition is the first one for CMC surfaces with $-1 < H < 1$ in H^3.

Isothermic discrete surfaces and their Christoffel transforms. Consider a discrete surface $\mathfrak{f}_p \in \mathrm{Im}H \approx R^3$, where p is any point in a discrete lattice domain (locally always

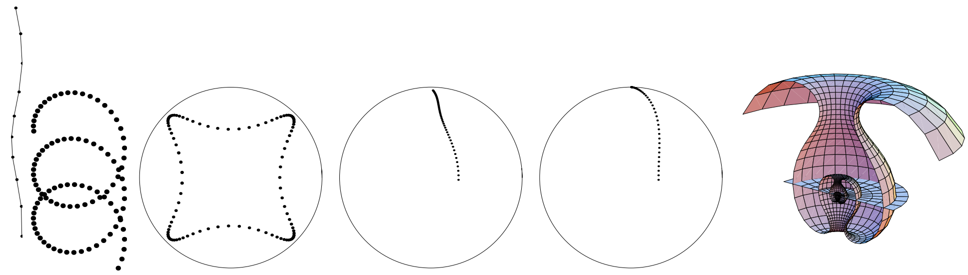


Figure 7: Discrete profile curves for discrete CMC surfaces of revolution. The meanings of these graphics are explained in Example 1.

a subdomain of Z^2). Consider one quadrilateral in the lattice with vertices p, q, r, s (for example the points (m,n), $(m+1,n)$, $(m+1,n+1)$, $(m,n+1)$ for some $m,n \in Z$) ordered counterclockwise about the quadrilateral. We define the cross ratio of this quadrilateral as

$$q_{pqrs} = (\mathfrak{f}_q - \mathfrak{f}_p)(\mathfrak{f}_r - \mathfrak{f}_q)^{-1}(\mathfrak{f}_s - \mathfrak{f}_r)(\mathfrak{f}_p - \mathfrak{f}_s)^{-1} .$$

When, for every quadrilateral, we can write the cross ratio as

$$q_{pqrs} = a_{pq}/a_{ps} \in R$$

so that the function a_{pq} defined on the edges of $\mathfrak{f}$ satisfies

$$a_{pq} = a_{sr} \in R \text{ and } a_{ps} = a_{qr} \in R ,$$

then we say that $\mathfrak{f}$ is *isothermic*. Note that the a_{pq} are symmetric, i.e. $a_{pq} = a_{qp}$ for any adjacent p and q.

Remark 1. *When considering a smooth surface $x(u,v)$ in R^3 and defining*

$$\mathfrak{Q} = (x(u+\epsilon, v-\epsilon) - x(u-\epsilon, v-\epsilon))(x(u+\epsilon, v+\epsilon) - x(u+\epsilon, v-\epsilon))^{-1} \times$$
$$(x(u-\epsilon, v+\epsilon) - x(u+\epsilon, v+\epsilon))(x(u-\epsilon, v-\epsilon) - x(u-\epsilon, v+\epsilon))^{-1} ,$$

Bobenko and Pinkall [3] proved that

$$\mathfrak{Q} = -I + \mathcal{O}(\epsilon)$$

if and only if x is conformal, and

$$\mathfrak{Q} = -I + \mathcal{O}(\epsilon^2)$$

if and only if x is isothermic. This leads to the following definition for discrete isothermic surfaces in the narrow sense: $\mathfrak{f}$ is discrete isothermic if

$$(\mathfrak{f}_q - \mathfrak{f}_p)(\mathfrak{f}_r - \mathfrak{f}_q)^{-1}(\mathfrak{f}_s - \mathfrak{f}_r)(\mathfrak{f}_p - \mathfrak{f}_s)^{-1} = -I$$

for all quadrilaterals. However, with this definition, transformations, such as the Calapso transform, of isothermic surfaces will not remain isothermic. Hence the broader definition given above has been taken up.

When $\mathfrak{f}$ is isothermic, we can define the Christoffel transform $\mathfrak{f}^*$ of $\mathfrak{f}$ by

$$d\mathfrak{f}^*_{pq} d\mathfrak{f}_{pq} = a_{pq} .$$

We can then prove the following:

Lemma 10. *[2] $\mathfrak{f}$ is isothermic if and only if there exists a discrete surface $\mathfrak{f}^*$ satisfying the above equation for $d\mathfrak{f}^*$, and $\mathfrak{f}^*$ is isothermic with the same cross ratios as $\mathfrak{f}$.*

Calapso transformations. Like in the smooth case, we can define Calapso transformations in the discrete case. For adjacent vertices p, q, we define T by

$$T_q = T_p(1 + \lambda\tau_{pq}) \ , \quad \tau_{pq} = c \begin{pmatrix} \mathfrak{f}_p \\ 1 \end{pmatrix} (\mathfrak{f}_q^* - \mathfrak{f}_p^*) \begin{pmatrix} 1 & -\mathfrak{f}_q \end{pmatrix} \ .$$

Lemma 11. *If $\mathfrak{f}$ is discrete isothermic, then T exists.*

Linear conserved quantities. We can now discretize (5) to obtain

$$(1 + \lambda\tau_{pq})(Q + \lambda Z)_q = (Q + \lambda Z)_p(1 + \lambda\tau_{pq}) \ , \tag{8}$$

where $\lambda \in R$ and $Q, Z \in R^{4,1}$ are functions on the lattice domain.

We can derive this discretization as follows: We say that $\mathfrak{f}$ is CMC if if there exists a $P = Q + \lambda Z$ so that TPT^{-1} is constant, just like in the smooth case. This is equivalent to

$$T_q P_q T_q^{-1} = T_p P_p T_p^{-1}$$

for all adjacent vertices p, q, which is equivalent to

$$(1 + \lambda\tau_{pq})P_q = P_p(1 + \lambda\tau_{pq}) \ ,$$

which becomes Equation (8) above.

We now come to our goal:

Definition 5. *If a linear conserved quantity $Q + \lambda Z$, $Q \neq 0$, exists for an isothermic discrete surface $\mathfrak{f}$, we say that $\mathfrak{f}$ is of constant mean curvature in the space form M determined by Q.*

Remark 2. *One can see, in the case that $M = R^3$, that the above definition is equivalent to the definition found by Bobenko and Pinkall [2]: $\mathfrak{f}$ is CMC if $|\mathfrak{f}_p - \mathfrak{f}_p^*|^2$ is constant, and then it is the constant H_0^{-2}. Also, the property of being discrete CMC is preserved by Calapso transformations, so the definition here is the right one for the space form $M_1 = S^3$, and also for the space form $M_{-1} = H^3$ when the mean curvature H_{-1} has absolute value at least 1.*

Furthermore, we can define the constant mean curvature of $\mathfrak{f}$ to be the lower left entry of Z, and the normal vector of $\mathfrak{f}$ to be the upper left entry of Z. Note that here the mean curvature and the normal vector will be unique if Z is.

Looking at the coefficients in front of the λ^k in Equation (8) for $k = 0, 1, 2$, we immediately have the following lemma:

Lemma 12. *Equation (8) is equivalent to $dQ = 0$ and $dZ_{pq} = Q_p\tau_{pq} - \tau_{pq}Q_q$ and $\tau_{pq}Z_q = Z_p\tau_{pq}$.*

Example 1. In the last figure, we show discrete CMC surfaces of revolution. The first two curves are profile curves for discrete nonminimal CMC surfaces of revolution in R^3, the first being unduloidal and the second nodoidal. (For each of these two curves, the axis of rotation producing the surface is a vertical line drawn to the left of the curve, and is not shown in the figure.) The third picture shows the profile curve for a discrete CMC surface of revolution in S^3, where S^3 is stereographically projected to R^3, and the shown circle is a geodesic of S^3 that is also the axis of the surface – and furthermore, this example has a periodicity that causes it to close on itself and form a torus. The final three pictures show discrete CMC surfaces of revolution in H^3. The first two, with $H > 1$ and $H = 1$ respectively, are shown in the Poincare model, and the first is unduloidal while the second looks similar to a smooth embedded catenoid cousin. (For these two curves, the corresponding axis of revolution is the vertical line between the uppermost and lowermost points of the circle shown, and this circle lies in the boundary sphere at infinity of H^3.) The last picture is a minimal surface that lies in both copies of $M_{-1} = H^3 \cup H^3$, and the horizontal plane shown here is the virtual boundary at infinity of two copies of the halfspace model for H^3. This example was not known before, because the notion of discrete CMC was not defined before in this case.

Darboux transformations. The Darboux transforms of discrete surfaces have similar enveloping properties to the case of smooth surfaces. In the discrete case, the eight vertices of two corresponding quadrilaterals (one on the original surface and the corresponding one on the Darboux transform) all lie in one sphere.

Polynomial conserved quantities. Equation (8) can be extended to define surfaces with polynomial conserved quantities, as follows:

$$(1+\lambda\tau_{pq})(Q+\lambda P_1+\lambda^2 P_2+...+\lambda^{n-1}P_{n-1}+\lambda^n Z)_q = \\ = (Q+\lambda P_1+\lambda^2 P_2+...+\lambda^{n-1}P_{n-1}+\lambda^n Z)_p(1+\lambda\tau_{pq})\ . \tag{9}$$

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