Fuchsian Diff Eq Data

Papers

Fuchsian differential equations of order $3, \ldots, 6$ *with three singular points and an accessory parameter*
Part I:Equations of order 6,
Part II:Equations of order 3,
Part III:Higher order versions of the Dotsenko-Fateev equation
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arXiv 2307.01358, math.CA Part I,
arXiv 2307.01360, math.CA Part II,
arXiv 2111.11192v3, math.CA Part III;
doi number will be listed later

Explanation of data

For the Fuchsian differential equations treated in the papers, we list the equations and the shift operators in the form that the data is readable by the help of the mathematical software Maple.

Definition of the differential equations

The equations we treated in the papers Part I, II and III are

$$\begin{array}{ll} \text{Part1} & H_j,\ G_j,\ E_j,\quad (j=6,5,4,3),\ \text{and } E_2 \\ \text{Part2} & H_3,\ E_3,\quad SE_3,\ Z_3,\ E_{3a},\cdots,E_{3d}, \\ \text{Part3} & SE_J,\ (j=6,5,4,3), \end{array}$$

where E_2 is the Gauss hypergeometric equation, which is related to all others; They are mutually related as in the following figure

$$\begin{array}{lcccccccl} & & & & & & & & \text{Part 3} \\ \text{Part 1} & H_6 & \longrightarrow & G_6 & \longrightarrow & E_6 & \longrightarrow & & SE_6 \\ & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ \text{Part 1} & H_5 & \longrightarrow & G_5 & \longrightarrow & E_5 & \longrightarrow & & SE_5 \\ & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ \text{Part 1} & H_4 & \longrightarrow & G_4 & \longrightarrow & E_4 & \longrightarrow & & SE_4 \\ & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ \text{Part 2} & H_3 & \longrightarrow & G_3 & \longrightarrow & E_3 & \longrightarrow & & SE_3,\ Z_3,\quad E_{3a},\cdots,E_{3d} \end{array}$$

Horizontal arrows stand for specializations keeping the spectral type, and vertical lines for factorizations. Every equation except E_2 has one accessory parameter.

In the following, we use a maple notation dx used in *DEtools* of Maple, which means the differential operator d/dx relative to the variable x and the notation z denotes the operator $x * dx$. The mark $*$ means the multiplication as well as composition of differential operators.

We summarize the defintion of the differential equations. The text-file **equations.txt** lists the maple forms of the differential equations.

- $H_6 = H_6(e1, \ldots, e9, T10) = T0 + T1 * dx + T2 * dx^2 + T3 * dx^3$

```
T0:=(z+s+2)*(z+s+1)*(z+s)*(z+e7)*(z+e8)*(z+e9):
T1:=(z+s+2)*(z+s+1)*B1:
T2:=(z+s+2)*B2:
T3:=-(z+3-e1)*(z+3-e2)*(z+3-e3):
B1:=T13*z^3+T12*z^2+T11*z+T10:
B2:=T23*z^3+T22*z^2+T21*z+T20:
```

where s is determined by the Fuchs relation $e1 + \cdots + e9 + 3s + 9 = 15$. The coefficients Tij, except $T10$ are polynomials in $e = (e1, \dots, e9)$, and $T10$ is the accessory parameter. $G_6 = G_6(e, a)$ is a specialization of H_6, where $T10$ is a polynomial $T10(e, a)$ in e with a set a of parameters. $E_6 = E_6(e) = G_6(e, 0)$. The operators $B1$, $B2$ and the polynomials Tij are given in the file **equations.txt**.

- SE_6 is a specialization of E_6 with the condition

$$e1 - 2e2 + e3 = e4 - 2e5 + e6 = e7 - 2e8 + e9.$$

It is parameterized by (a, b, c, g, p, q, r) by the relations

```
e1:=p+r+1:
e2:=a+c+p+r+2:
e3:=2a+2c+g+p+r+3:
e4:=q+r+1:
e5:=b+c+q+r+2:
e6:=2b+2c+g+q+r+3:
e7:=-2c-p-q-r-1:
e8:=-a-b-2c-p-q-r-g-2:
e9:=-2a-2b-2c-p-q-r-g-3:
s:=-r:
```

The equation H_5 ($j = 5, 4, 3$) has one accessory parameter. $G_j(e, a)$ ($j = 5, 4, 3$) is defined as above, and E_j ($j = 5, 4, 3$) is defined by $G_j(e, 0)$ ($j = 5, 4, 3$). In the following we tabulate only E_j and SE_j ($j = 5, 4, 3$).

- The equation $E_5 = E_5(e1, \dots, e8) = x * Pn + P0 + P1 * dx + P2 * dx^2$ is the quotient of E_6 with the condition $e9 = 0$ divied by dx on the right.

```
Pn:=(z-r+1)*(z-r+2)*(z-r+3)*(z+e7+1)*(z+e8+1):
P0:=(z-r+1)*(z-r+2)*B1(e9=0):
P1:=(z-r+2)*B2(e9=0):
P2:=-(z+3-e1)*(z+3-e2)*(z+3-e3):
```

where $r = (e1 + \cdots + e8 - 6)/3$.

- SE_5 is a specialization of E_5 with the condition

$$e1 - 2e2 + e3 = e4 - 2e5 + e6 = e7 - 2e8.$$

It is parameterized by (a, b, c, g, p, q) as

```
e1:= -2*a - 2*b - 2*c - g - q - 2:
e2:= -a - 2*b - c - g - q - 1:
e3:= -2*b - q:
e4:= -2*a - 2*b - 2*c - g - p - 2:
e5:= -b  - 2*a- c - g - p - 1:
e6:= -2*a - p:
e7:= 2*a + 2*b + g + 2:
e8:= a + b + 1:
```

- $E_4 = E_4(e1, \dots, e7) = Q_0 + Q_1 * dx + Q_2 * dx^2$ is defined as

```
Q0:=(z+e5)*(z+e6)*(z+e7)*(z+e8):
Q1:=-2*z^3+Q12*z^2+Q11*z+Q10:
Q12:=e1+e2-e5-e6-e7-e8-5:
Q11:=3*(e1+e2)-e1*e2+e3*e4-e5*e6-e5*e7-e5*e8-e6*e7-e6*e8-e7*e8-8:
Q2:=(z-e1+2)*(z-e2+2):
```

where $e8$ is determined by the Fuchs raltion $e1 + e2 + \cdots + e7 + e8 = 4$ and $Q10$ is given in **equations.txt**. The equation is written also as

$$E_4 := x^2 * (x-1)^2 dx^4 + p3 * dx^3 + p2 * dx^2 + p1 * dx + p0 :$$

where

```
p3:= x*(x-1)*((-t11-t12+10)*x+t11-5):
p2:= (-3*t11-3*t12+t23+19)*x^2 +(5*t11+t12-t21+t22-t23-19)*x+4-2*t11+t21:
p1:= (t3-t11-t12+t23+5)*x+Q10:
p0:= e5*e6*e7*e8:
```

Refer to **equations.txt** for $t11$, $t12$, ...

- SE_4 is a specialization of E_4 with the condition

$$e1 - 2e2 + 1 = e3 - 2e4 + 1 = e5 - 2e6 + e7 + e8, \quad e1 + \cdots + e7 + e8 = 4.$$

It is parameterized by (a, b, c, g, u) as

```
e1:=2+2a+2c+g-u:
e2:=1+a+c-u:
e3:=2+2b+2c+g-u:
e4:=1+b+c-u:
e5:=u+1:
e6:=-1-a-2c-g-b+u:
e7:=-2c+u:
e8:=-2-2a-2b-2c-g+u:
```

- Z_4 is a specialization of E_4 with the condition

$$e1 + e2 - 1 = e3 + e4 - 1 = -(e5 + e6 - 1).$$

It is parameterized by $(A0, A1, A2, A3, k)$ as

```
e1=1/2-A0-k:
e2=1/2+A0-k:
e3=1/2-A1-k:
e4=1/2+A1-k:
e5=1/2-A2+k:
e6=1/2+A2+k:
e7=1/2-A3+k:
e8=1/2+A3+k:
```

- $\mathrm{ST}_4 = \mathrm{ST}_4(e1, \ldots, e6) = V0 + V1 * dx + V2 * dx^2$ is given as

```
V0 := (z+s+1)*(z+s)*(z+e5)*(z+e6):
V1 := (z+s+1)*(-2*z^2+(e1+e2-e5-e6-4)*z
        +1/4*((e6-e5)^2-(e3-e4)^2+(e1-e2)^2+2*(e1+e2-3)*(e5+e6+1)-1)):
V2 := (z+2-e1)*(z+2-e2):
```

where s is determined by the Fuchs relation $e1 + e2 + e3 + e4 + e5 + e6 + 2 * s + 3 = 6$.

- ${}_4E_3(a0, a1, a2, a3; b1, b2, b3) := (z+a0)*(z+a1)*(z+a2)*(z+a3) - (z+b1)*(z+b2)*(z+b3)*dx$ is the generalized hypergeometric equation of rank 4.

- $E_3 = E_3(e1, \ldots, e6) = x * Sn + S0 + S1 * dx$ is defined as

```
Sn:=(z+e5)*(z+e6)*(z+e7):
S0:= -2*z^3+(2*e1+2*e2+e3+e4-3)*z^2
   +(-e1*e2+(e3-1)*(e4-1)-e5*e6-(e5+e6)*e7)*z+a00,\\
S1:=(z-e1+1)*(z-e2+1):
```

where $e7$ is determined by the relation $e1+e2+e3+e4+e5+e6+e7=3$ and $a00$ is the accessory parameter defined as

```
54*a00:= -4*(e1+e2-e3-e4)^3-27*e5*e6*e7+9*(e1+e2-e3-e4)*(e5*e6+e5*e7+e6*e7-2)
    +9*e1*e2*(e1+e2-1)+18*(e1+e2-1)*(e3^2+e3*e4+e4^2)
    -9*e3*e4*(e3+e4-1)-18*(e3+e4-1)*(e1^2+e1*e2+e2^2):
```

- SE_3 is a specialization of E_3 with the condition

$$2e1-e2=2e3-e4=-e5+2e6-e7.$$

It is parameterized by (a,b,c,g) as:

```
e1:=a+c+1:
e2:=2e1+g:
e3:=b+c+1:
e4:=2e3+g:
e5:=-2c:
e6:=-(a+b+2c+g+1):
e7:=2e6+g-e5:
```

The accessory parameter turns out to be

$$a00=c*(2*a+2*c+1+g)*(2*a+2*b+2*c+2+g).$$

- Z_3 is a specialization of E_3 with the condition

$$e1+e2+e5=e3+e4+e5=1.$$

It is parameterized by $(A0,A1,A2,A3)$ as

```
e1:=-A0-A3:
e2:=A0-A3:
e3:=-A1-A3:
e4:=A1-A3:
e5:=2*A3+1:
e6:=A2+A3+1:
```

The accessory parameter is given by

$$a00:=(2*A3+1)*(A0^2-A1^2-A3^2+A2^2-2*A3-1)/2.$$

- E_{3a}, E_{3b}, E_{3c}, E_{3d} are specializations of E_3 with the conditions

$$e3=e1, e4=e2;$$
$$e2=-e3-e5-2*e6+3-e1, e4=e3-e5+e6;$$
$$e2=2*e1+e3+e4, e5=1-e1-e3-e4;$$
$$e2=3/2-e1-1/2*e3-1/2*e4-3/2*e6, e5=e1+e6;$$

respectively.

- ${}_3E_2(a0,a1,a2;b1,b2):=(z+a0)*(z+a1)*(z+a2)-(z+b1)*(z+b2)*dx$ is the generalized hypergeometric equation of rank 3.

- $E_2:=E_2(e1,e2,e3,e4)=E(a,b,c)=(z+a)*(z+b)-(z+c)*dx$ is the Gauss hypergeometric equation: We used the parameters $(e1,e2,e3,e4)$ defined as $e1=1-c$, $e2=c-a-b$, $e3=a$, and $e4=b$, where $e1+e2+e3+e4=1$.

Shift operators

We review definitions of shift operators and explain the text-files for such operators.

Given a differential operator $E(a)$ with parameter a of order n, suppose a shift operator P_{a+} (*resp.* P_{a-}) exists, which is an operator of order lower than n sending $\mathrm{Sol}(E(a))$ to $\mathrm{Sol}(E(a+1))$ (*resp.* $\mathrm{Sol}(E(a-1))$), we have a shift relation such as

$$E(a+1) \circ P_{a+}(a) = Q_{a+}(a) \circ E(a) \quad (\textit{resp. } E(a-1) \circ P_{a-}(a) = Q_{a-}(a) \circ E(a)),$$

where $Q_{a\pm}$ are some operators of the same order of $P_{a\pm}$.

In the following, we list the operators $P_{a\pm}$ and $Q_{a\pm}$ for each equation and each shift.

For the Gauss equation E_2, the following shift operators are classically known:

$$\begin{array}{rclrcl} P_{a+} &=& x*dx+a, & Q_{a+} &=& x*dx+a+1, \\ P_{a-} &=& x(x-1)*dx+a+bx-c, & Q_{a-} &=& x(x-1)*dx+a+bx-c+x-1, \\ P_{c+} &=& (x-1)*dx+a+b-c, & Q_{c+} &=& P_{c+}, \\ P_{c-} &=& x*dx+c-1, & Q_{c-} &=& P_{c-}. \end{array}$$

For the equations E_4, we find only a simple shift operator, which are mentioned in the paper. For the equation E_3, we could not find a shift operator.

For other equations, refer to the list of shift operators given in

G6PQ.txt, G6memo.txt	for G6,
E6PQ.txt	for E6,
SE6PQ.txt	for SE6,
Z6PQ1.txt, Z6PQ2.txt	for Z6,
E5PQ.txt	for E5,
SE5PQ.txt	for SE5,
SE4PQ.txt	for SE4,
Z4PQ.txt	for Z4,
ST4PQ.txt	for ST4,
SE3PQ.txt	for SE3,
Z3PQ.txt	for Z3,
E3aPQ.txt	for E3a,
E3bPQ.txt	for E3b,
E3cPQ.txt	for E3c,
E3dPQ.txt	for E3d.

The equations Z_4 and Z_6 are codimension-2 subfamilies of E_4 and E_6; they are connencted to Z_3 via addition and middle convolution. The equations $E_{3a}, \ldots, E_{3d}$ are specializations of E_3 treated in Part 2, which admits shift operators.